UNIT - 2 SIMPLEX METHOD

It is not possible to obtain the graphical solution to the L.P. problem having more than two variables. In problems with more than two variables, Simplex method is used to solve the problem.

The simplex method provides a systematic algorithm which consists of moving from one basic feasible solution (one vertex) to another in a prescribed manner so that the value of the objective function is improved. This procedure of jumping from vertex to vertex is repeated. If the objective function is improved at each jump, then no basis can ever repeat and there is no need to go back to vertex already covered. Since the number of vertices are finite, the process must lead to the optimal vertex in a finite number of steps.

SLACK VARIABLE :

A non-negative variable which is added to the left hand side of the constraint to convert it into an equation is called a slack variable.

For example: If a constraint has ≤ sign, then in order to make it an equation add something positive to the left hand side.

$3x_1 + x_2 \leq 2, \quad 2x_1 + 4x_2 \leq 5, \quad x_1, x_2 \geq 0$

Add the slack variables $S_1 \geq 0$, $S_2 \geq 0$ on the LHS of above inequalities respectively.

$\therefore \; 3x_1 + x_2 + S_1 = 2$

$2x_1 + 4x_2 + S_2 = 5$

$x_1, x_2, S_1, S_2 \geq 0$

SURPLUS VARIABLE:

A positive variable which is subtracted from the left hand side of the constraint to convert it into an equation is called a surplus variable.

For example: If a constraint has $\geq$ sign, then in order to make it an equation subtract something non-negative from its left hand side.

$3x_1 + x_2 \geq 2, \quad 2x_1 + 4x_2 \geq 5$ and $x_1, x_2 \geq 0$.

Subtract the surplus variables $S_3 \geq 0$, $S_4 \geq 0$ from the LHS of above inequalities respectively.

$\therefore \; 3x_1 + x_2 - S_3 = 2$

$2x_1 + 4x_2 - S_4 = 5$

$x_1, x_2, S_3, S_4 \geq 0$

Standard form of L.P.P. for applying Simplex method

1. All the constraints should be converted into equations except for the non-negativity restrictions which remain as inequalities (≥ 0)

For example: $3x_1 - 4x_2 \geq 9$, $x_1 + 2x_2 \leq 3$

$$3x_1 - 4x_2 - x_3 = 9, \quad x_1 + 2x_2 + x_4 = 3$$

and $x_3 \geq 0$, $x_4 \geq 0$

2. The right side element of each constraint should be made non-negative (if not).

$\Rightarrow$ It can be made positive on multiplying both sides of the resulting equation by (-1).

For example: $3x_1 - 4x_2 \geq -6$

$$\Rightarrow 3x_1 - 4x_2 - x_3 = -6$$

x^{ly} by $(-1) \Rightarrow$ $-3x_1 + 4x_2 + x_3 = 6$

3. All variables must have non-negative values.

$\Rightarrow$ A variable which is unrestricted in sign (that is positive, negative, or zero) is equivalent to the difference between two non-negative variables. Thus, if x is unconstrained in sign, it can be replaced by $(x' - x'')$ where $x' - x''$ are both non-negative

i.e., $x' \geq 0$, $x'' \geq 0$.

(4) The objective function should be of maximization from.

⇒ The minimization of a function $f(x)$ is equivalent to the maximization of the negative expression of this function $f(x)$ i.e.,

$$\min f(x) = \max[-f(x)]$$

For example : min. $Z = 3x_1 + 4x_2 + 6x_3$

is equivalent to max$(-Z)$.

i.e., max $Z' = -3x_1 - 4x_2 - 6x_3$; where $Z' = -Z$

Important Definitions :

(1) Solution to LPP: Any set $x = \{x_1, x_2, \dots x_{n+m}\}$ of variables is called a solution to L.P.P. if it satisfies only the set of given constraint equations. m - no of constraints, n - no of variables

(2) Feasible Solution (F.S): Any set $x = \{x_1, x_2, \dots x_{n+m}\}$ of variables is called a feasible solution of L.P.P. if it satisfies the given set of constraint equations and also non-negativity restrictions also.

(3) Basic Solution (BS): A basic solution to the set of constraints is a solution obtained

by setting any n variables (among $m+n$ variables) equal to zero and solving for remaining m variables, provided the determinant of the coefficients of these m variables is non-zero. Such m variables are called basic variables and remaining n zero-valued variables are called non-basic variables.

(4) Basic Feasible Solution (BFS): A basic feasible solution is a basic solution which also satisfies the non-negativity restriction. i.e., all basic variables are non-negative.

Basic feasible solutions are of two types.

(a) Non-degenerate BFS: A non-degenerate BFS is the BFS which has exactly m positive x_i $(i = 1, 2, \ldots m)$. In other words, all m basic variables are positive, and the remaining n variables will be all zero.

(b) Degenerate BFS: A BFS is called degenerate, if one or more basic variables are zero valued.

(5) Optimum Basic Feasible Solution: A BFS is said to be optimum, if it also optimizes (maximizes or minimizes) the objective function.

(6) Unbounded Solution: If the value of the objective function Z can be increased or decreased indefinitely, then such solutions are called unbounded solutions.

(7) Alternative or multiple optimal solutions: In some linear programming problems, there may exist more than one feasible solution such that their objective function values are equal to the optimal value of the problem. In such cases, all these feasible solutions are optimal solutions and the problem is said to have alternative or multiple optimal solutions.

Computational Procedure of Simplex Method

The computational procedure of the simplex method is explained by the following example

Problem 1:

maximize $Z = 2x_1 + 3x_2$,

StC $2x_1 + x_2 \leq 12$

$x_1 + 3x_2 \leq 15$

$x_1, x_2 \geq 0$

Step 1: All the right side constants of the constraints must be non-negative. If not, it should be changed to positive value on multiplying both sides of the constraints by -1.

Step 2: convert the inequality constraints to equations by introducing the non-negative slack or surplus variables. The coefficients of slack or surplus variables are always taken zero in the objective function. In the above problem, all inequality constraints being "≤", only slack variables s_1 and s_2 are introduced.

$$\therefore \max Z = 2x_1 + 3x_2 + 0s_1 + 0s_2$$

stc

$$2x_1 + x_2 + s_1 = 12$$

$$x_1 + 3x_2 + s_2 = 15$$

$$x_1, x_2, s_1, s_2 \geq 0.$$

Step 3: Construct the starting simplex table as follows:

			2	3	0	0	← obj row
			x_1	x_2	s_1	s_2	
0	s_1	12	2	1	1	0	12
0	s_2	15	1	3	0	1	5
	Z	0	-2	-3	0	0	

Constant column

Variable column

objective column

Identity matrix

Key row

Coefficient matrix

Index row / Net evaluation row

Key column

Step 4: Index row number $= \Sigma$ [No. in the column × corresponding No. in the obj column] − No in the obj row.

If all the index row numbers are ≥ 0 the solution under test will be optimal. If any one of the index row number is negative, the solution under test is not optimal. Then to improve the solution proceed to the next step.

If any index row number is negative and consequently if all the elements in that column are negative or zero then the solution under test will be unbounded.

Step 5: Fix the key column by choosing the most −ve in the index row.

Step 6: Fix the key row by choosing the minimum obtained by dividing the constant column No. by the corresponding key column No. (* only +ve values should be considered)

Key No: is the one common to both the key row and the key column.

Step 7: Main row = $\frac{\text{Key row}}{\text{Key No.}}$

Step 8:

New No. = old No. − [Corresponding key row No. × Corresponding key Column No.] / Key No.

Step 9: If all the index numbers in an index row or net evaluation row are positive then the solution is optimum. otherwise repeat the steps 5, 6, 7 and 8 till all the index numbers become positive.

			↓				
			2	3	0	0	
			x_1	x_2	S_1	S_2	
0	S_1	12	2	1	1	0	12
0	S_2	15	1	3	0	1	5 →
	Z	0	−2	−3	0	0	
	S_1	7	5/3 ↓	0	1	−1/3	21/5 = 4.2
main row →	x_2	5	1/3	1	0	1/3	15
	Z	15	−1	0	0	1	
	x_1	21/5	1	0	3/5	−1/5	
	x_2	18/5	0	1	−1/5	6/15	
	Z	96/5	0	0	3/5	4/5	← Index row

Since all the Index numbers are +ve in index row, the solution is optimum.

$\therefore x_1 = \frac{21}{5}, \quad x_2 = \frac{18}{5}$

$Z = 2\left(\frac{21}{5}\right) + 3\left(\frac{18}{5}\right) = \frac{42+54}{5}$

$\therefore Z = \frac{96}{5}$

Problem 2: min $Z = x_1 - 3x_2 + 2x_3$

s.t.c

$3x_1 - x_2 + 3x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1; x_2, x_3 \geq 0$

Soln:- Converting the objective function from minimization to maximization.

max $-Z = -x_1 + 3x_2 - 2x_3 = \max Z'$

where $-Z = Z'$

$3x_1 - x_2 + 3x_3 + S_1 = 7$

$-2x_1 + 4x_2 + S_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

$\therefore \max Z' = -x_1 + 3x_2 - 2x_3 + 0S_1 + 0S_2 + 0S_3$

			-1	3 ↓	-2	0	0	0	
			x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	7	3	-1	3	1	0	0	-7
0	s_2	12	-2	4	0	0	1	0	3 → Mi
0	s_3	10	-4	3	8	0	0	1	10/3
	z'	0	1	-3	2	0	0	0	
	s_1	10	5/2	0	3	1	1/4	0	4 →
	x_2	3	-1/2	1	0	0	1/4	0	-ve
	s_3	1	-5/2	0	8	0	-3/4	1	-ve
	z'	9	-1/2	0	2	0	3/4	0	
	x_1	4	1	0	6/5	2/5	1/10	0	
	x_2	5	0	1	3/5	1/5	3/10	0	
	s_3	11	0	0	11	1	-1/2	1	
	z'	11	0	0	13/5	1/5	4/5	0	

Since all the index numbers are +ve the solution is optimum.

$\therefore x_1 = 4, x_2 = 5, x_3 = 0$

$z' = 11$

$\therefore z = -11$

Problem 3: Max $Z = 2x_1 + 2x_2 + 4x_3$.

s.t.c

$2x_1 + 3x_2 + x_3 \leq 240$

$x_1 + x_2 + 3x_3 \leq 300$

$x_1 + 3x_2 + x_3 \leq 300$

$x_1, x_2, x_3 \geq 0$

Solution: $2x_1 + 3x_2 + x_3 + S_1 = 240$

$x_1 + x_2 + 3x_3 + S_2 = 300$

$x_1 + 3x_2 + x_3 + S_3 = 300$

$Z = 2x_1 + 2x_2 + 4x_3 + 0S_1 + 0S_2 + 0S_3$

			2	2	4 ↓	0	0	0	
			x_1	x_2	x_3	S_1	S_2	S_3	
0	S_1	240	2	3	1	1	0	0	240
0	S_2	300	1	1	3	0	1	0	100 min →
0	S_3	300	1	3	1	0	0	1	300
	Z	0	-2	-2	-4	0	0	0	
	S_1	140	5/3	8/3	0	1	-1/3	0	84 min →
	x_3	100	1/3	1/3	1	0	1/3	0	300
	S_3	200	2/3	8/3	0	0	-1/3	1	300
	Z	400	-2/3	-2/3	0	0	4/3	0	
	x_1	84	1	8/5	0	3/5	-1/5	0	
	x_3	72	0	-1/5	1	-1/5	6/15	0	
	S_3	144	0	24/15	0	-2/5	-1/15	1	
	Z	456	0	6/15	0	2/5	18/15	0	

This became → x_3 not x_2

* Note: Since $-2/3$ and $-2/3$ exists at two places, choose any one and find the key row then find the value of Z. Again choose another $-2/3$ and find Z, then select the column which resulted in maximum value of Z as the key column.

i.e., for one column the value of $Z = 400 - \frac{140(-2/3)}{5/3}$

$= 456$.

For another column the value of $Z = 400 - \frac{140(-2/3)}{8/3}$

$= 435$.

Therefore the first column which resulted in maximum value of Z was selected as key column.

Further since all the index numbers are +ve, the solution is optimum.

$\therefore x_1 = 84, \; x_2 = 0, \; x_3 = 72$

And $Z = 2(84) + 0 + 4(72)$

$\therefore Z = 456$

Problem 4: Max $Z = 107x_1 + x_2 + 2x_3$

Stc

$14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + \frac{1}{2}x_2 - 6x_3 \leq 5$

$3x_1 - x_2 - x_3 \leq 0$

$x_1, x_2, x_3, x_4 \geq 0$

Solution : Divide the first constraint equation by 3 (co-efficient of x_4) and then treat x_4 as the slack variable.

$$\therefore \quad \frac{14}{3} x_1 + \frac{1}{3} x_2 - 2x_3 + x_4 = \frac{7}{3}$$

$$16 x_1 + \frac{1}{2} x_2 - 6x_3 + x_5 = 5$$

$$3x_1 - x_2 - x_3 + x_6 = 0$$

where, $x_5 \geq 0$ and $x_6 \geq 0$ are slack variables.

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

$$Z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$$

			107	1	2	0	0	0	
			x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	7/3	14/3	1/3	-2	1	0	0	1/2
0	x_5	5	16	1/2	-6	0	1	0	5/16
0	x_6	0	3	-1	-1	0	0	1	0/3 → min
	Z	0	-107	-1	-2	0	0	0	
	x_4	7/3	0	17/9	-4/9	1	0	-14/9	-ve
	x_5	5	0	35/6	-2/3	0	1	-16/3	-ve
	x_1	0	1	-1/3	-1/3	0	0	1/3	-ve
	Z	0	0	-110/3	-113/3	0	0	107/3	
				= -36.67	-37.66				

The most -ve column is x_3 and to fix the key row it is not possible because all are -ve. This indicates that there is an unbounded solution to the given L.P.P.

Problem 5: Show that the LPP

$$\max Z = 4x_1 + x_2 + 3x_3 + 5x_4$$

StC $\quad 4x_1 - 6x_2 - 5x_3 + 4x_4 \geq -20$

$3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$

$8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$

$x_1, x_2, x_3, x_4 \geq 0$ has an unbounded soln.

Soln: Transforming the given inequality constraints into equations.

$4x_1 - 6x_2 - 5x_3 + 4x_4 - S_1 = -20$; S_1 is surplus variable.

multiplying the above equation by

$(-1) \Rightarrow -4x_1 + 6x_2 + 5x_3 - 4x_4 + S_1 = 20$

$3x_1 - 2x_2 + 4x_3 + x_4 + S_2 = 10$

$8x_1 - 3x_2 + 3x_3 + 2x_4 + S_3 = 20$

$x_1, x_2, x_3, x_4, S_1, S_2, S_3 \geq 0$

And $Z = 4x_1 + x_2 + 3x_3 + 5x_4 + 0S_1 + 0S_2 + 0S_3$

Solution to LP problem by BIG-M-method (Charnes Penalty method)

Computational steps of Big-M-method is as follows.

Step (1) Express the problem in the standard LPP form.

Step (2) If the constraints are ≤ add slack variables only on the left hand side of the constraint equation.

Step (3) If the constraints are ≥ subtract the surplus variable and add the artificial variable on the left hand side of the constraints.

Step (4) If the constraints are = add artificial variables only on the left hand side of the constraint.

The main purpose of introducing the artificial variables is to get the initial basic feasible solution.

Step (5) The coefficients of the artificial variables in the objective row is −M, where M is a huge value unspecified but sufficiently large.

Step (6) In the initial variable column write only the slack and artificial variables.

Step (7) Then follow the steps discussed in the simplex method.

Problem 1: Solve the following L.P.P. by Big-M method.

$$\text{minimize } Z = 2x_1 + x_2$$

s.t.c

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solution: Converting the objective function from minimization to maximization

i.e., $\text{mini } Z = \max(-Z) = \max Z'$

$\therefore \max Z' = -2x_1 - x_2$

$3x_1 + x_2 + a_1 = 3$; a_1 is artificial variable

$4x_1 + 3x_2 - S_1 + a_2 = 6$; S_1 is surplus variable

$x_1 + 2x_2 + S_2 = 4$; S_2 is slack variable.

And $\max Z = -2x_1 - x_2 + 0S_1 + 0S_2 - Ma_1 - Ma_2$

* Refer next page for calculation.

From the solution, since all the index numbers are +ve the solution is optimum.

$\therefore x_1 = \frac{3}{5}, \; x_2 = \frac{6}{5}$

$\therefore Z = -2\left(\frac{3}{5}\right) - \left(\frac{6}{5}\right) = -\frac{6-6}{5} = -\frac{12}{5}$

$\therefore Z' = -\frac{12}{5}$ $\therefore \text{mini } Z = \frac{12}{5}$

↓ X_K

			-2	-1	0	0	-m	-m	
		X_B ↑	x_1	x_2	S_1	S_2	a_1	a_2	$\frac{X_B}{X_K}$ (where $X_K > 0$)
	S_2	4	1	2	0	1	0	0	4/1
-m	a_1	3	3	1	0	0	1	0	3/3 = 1 → min
-m	a_2	6	4	3	-1	0	0	1	6/4 = 3/2 = 1.5
	Z'	-9m	-7m+2	-4m+1	m	0	0	0	
	S_2	3	0	5/3	0	1	-1/3	0	9/5 = 1.8
	x_1	1	1	1/3	0	0	1/3	0	3/1 = 3
	a_2	2	0	5/3	-1	0	-4/3	1	6/5 = 1.2 →
	Z'	-2m-2	0	$\frac{(-5m+1)}{3}$	m	0	$\frac{(7m-2)}{3}$	0	
	S_2	1	0	0	+1	1	1	-1	
	x_1	3/5	1	0	1/5	0	3/5	-1/5	
	x_2	6/5	0	1	-3/5	0	-4/5	3/5	
	Z'	-12/5	0	0	1/5	0	$\frac{15m-6}{15}$	$\frac{5m-1}{5}$	

Problem 2:

Use penalty (Big-M) method to maximize:

$Z = 3x_1 - x_2$

stc $2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 3$

$x_2 \leq 4$

$x_1, x_2 \geq 0$

Soln: Transforming the given inequality constraints into equations

$2x_1 + x_2 - S_1 + a_1 = 2$; S_1 is surplus variable

$x_1 + 3x_2 + S_2 = 3$; S_2 is slack variable

$x_2 + S_3 = 4$; S_3 is slack variable

$x_1, x_2, S_1, S_2, S_3, a_1 \geq 0$.

and $Z = 3x_1 - x_2 + 0S_1 + 0S_2 + 0S_3 - Ma_1$

consider only slack and artificial variables for simplex calculation

Problem 3: min $Z = -8x_2$

stc $x_1 - x_2 \geq 0$

$2x_1 + 3x_2 \leq -6$

and x_1, x_2 are unrestricted.

Soln: Converting the objective function from minimization to maximization, we get

$\max Z' = 8x_2$

In this problem, the variables x_1 & x_2 are unrestricted in sign, ie, x_1 & x_2 may be +ve, -ve or zero. But the simplex method can be used only when the variable are non-negative (≥ 0). This difficulty can be removed by using the transformation.

$x_1 = x_1' - x_1''$ & $x_2 = x_2' - x_2''$ such that

$x_1' \geq 0,\ x_1'' \geq 0,\ x_2' \geq 0,\ x_2'' \geq 0$

∴ The given problem becomes

$\max\ Z = 8x_2' - 8x_2''$

stc $(x_1' - x_1'') - (x_2' - x_2'') \geq 0$

$-2(x_1' - x_1'') - 3(x_2' - x_2'') \geq 6$

$x_1', x_1'', x_2', x_2'' \geq 0$

Now introducing the surplus variables, and artificial variables.

$x_1' - x_1'' - x_2' + x_2'' - s_1 + a_1 = 0$; s_1 is surplus variable.

$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' - s_2 + a_2 = 6$; s_2 is surplus variable

$x_1', x_1'', x_2', x_2'', s_1, s_2, a_1, a_2 \geq 0$

$\max\ Z' = 0x_1' + 0x_1'' + 8x_2' - 8x_2'' + 0s_1 + 0s_2 - Ma_1 - Ma_2$

C_B	Basis	Sol.	x_1' (0)	x_1'' (0)	x_2' (8)	x_2'' (-8)	S_1 (0)	S_2 (0)	a_1 (-M)	a_2 (-M)	Ratio
-M	a_1	0	1	-1	-1	1	-1	0	1	0	0 → min
-M	a_2	6	-2	2	-3	3	0	-1	0	1	2
	z'	-6M	M	-M	4M-8	-4M+8	M	M	0	0	
	x_2''	0	1	-1	-1	1	-1	0	1	0	-ve
	a_2	6	-5	5	0	0	3	-1	-3	1	6/5 → min
	z'	-6M	5M-8	-5M+8	0	0	-3M+8	M	4M-8	0	
	x_2''	6/5	0	0	-1	1	-2/5	-1/5	2/5	1/5	
	x_1''	6/5	-1	1	0	0	3/5	-1/5	-3/5	1/5	
	z'	-48/5	0	0	0	0	16/5	8/5	$M-\frac{16}{5}$	$M-\frac{8}{5}$	

Since all the index numbers are +ve the solution is optimum.

$\therefore x_1' = 0, \; x_1'' = \frac{6}{5}, \; x_2' = 0, \; x_2'' = \frac{6}{5}$

Since $x_1 = (x_1' - x_1'')$ & $x_2 = (x_2' - x_2'')$

transforming the solution to original variables

$x_1 = 0 - \frac{6}{5} \quad \therefore x_1 = -\frac{6}{5}$

$x_2 = 0 - \frac{6}{5} \quad \therefore x_2 = -\frac{6}{5}$

and $\max z' = 8\left(-\frac{6}{5}\right) \quad \therefore \max z' = -\frac{48}{5}$

$\therefore \min z = \frac{48}{5}$ //

Problem 4

$$\max Z = 4x_1 + 5x_2 - 3x_3 + 50$$

Stc

$$x_1 + x_2 + x_3 = 10$$

$$x_1 - x_2 \geq 1$$

$$2x_1 + 3x_2 + x_3 \leq 40$$

$$x_1, x_2, x_3 \geq 0$$

Solution: If any constant is present in the obj fn, it should be deleted in the beginning and finally adjusted in optimum value of Z. If there is an equality in the constraints, then one variable can be eliminated from the inequalities with $\leq$ or $\geq$ sign.

∴ Subtracting constraint (i) from (iii) with a view to eliminate x_3 from (iii) and retaining x_3 in (i) to work as a slack variable.

∴ $x_1 + x_2 + x_3 = 10$, $x_1 - x_2 \geq 1$, $x_1 + 2x_2 \leq 30$

& $x_1, x_2, x_3 \geq 0$

∴ $x_1 + x_2 + x_3 = 10$; $x_3 \Rightarrow$ slack variable

$x_1 - x_2 - x_4 + a_1 = 1$; $x_4 \Rightarrow$ surplus variable

$x_1 + 2x_2 + x_5 = 30$; $x_5 \Rightarrow$ slack variable.

$x_1, x_2, x_3, S_1, S_2, a_1 \geq 0$

and $\max Z = 4x_1 + 5x_2 - 3x_3 + 0x_4 - Ma_1 + 0x_5$

C_B	Basis	X_B	x_1	x_2	x_3	x_4	a_1	x_5	Ratio
		C_j	4	5	-3	0	-M	0	
-3	x_3	10	1	1	1	0	0	0	10/1
-M	a_1	1	1	-1	0	-1	1	0	1/1 →
0	x_5	30	1	2	0	0	0	1	30/1
	Z	-30-M	-7-M	-8+M	0	M	0	0	

Basis	X_B	x_1	x_2	x_3	x_4	a_1	x_5	Ratio
x_3	9	0	2	1	1	–	0	9/2 →
x_1	1	1	-1	0	-1	1	0	-ve
x_5	29	0	3	0	1	–	1	29/3
Z	-23	0	-15	0	-7	7+M	0	

Basis	X_B	x_1	x_2	x_3	x_4	a_1	x_5
x_2	9/2	0	1	1/2	1/2	–	0
x_1	11/2	1	0	1/2	-1/2	–	0
x_5	31/2	0	0	-3/2	-1/2	–	1
Z	89/2	0	0	15/2	1/2	$\frac{29}{2}$+M	0

$\therefore x_1 = \frac{11}{2}, \; x_2 = \frac{9}{2}, \; x_3 = 0$

$\therefore \max Z = \frac{89}{2} + 50 = \frac{189}{2}$

from obj fn $Z = 4x_1 + 5x_2 - 3x_3 + 50$

$= 4 \times \frac{11}{2} + 5 \times \frac{9}{2} - 3(0) + 50$

$= \frac{189}{2}$

Problem 5: Max $Z = 3x_1 + 2x_2$

s.t.c $2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$.

Solution: Transforming the given constraints into equations.

$2x_1 + x_2 + S_1 = 2$; $S_1 \rightarrow$ Slack variable

$3x_1 + 4x_2 - S_2 + a_1 = 12$; S_2 - surplus variable.

$x_1, x_2, S_1, S_2, a_1 \geq 0$; $a_1 \rightarrow$ artificial variable

and max $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - Ma_1$

			3	2	0	0	-M	
			x_1	x_2	S_1	S_2	a_1	
0	S_1	2	2	1	1	0	0	2 →
-M	a_1	12	3	4	0	-1	1	3
	Z	-12M	-3M-3	-4M-2	0	M	0	
	x_2	2	2	1	1	0	0	
	a_1	4	-5	0	-4	-1	1	
	Z	-4M+4	5M+1	0	4M+2	M	0	

Here all the index row numbers are positive and the value of Z is in terms of M. Thus the given LPP does not possess an optimum basic feasible solution and there exists a pseudo-optimum solution

Big-M method **Problem 2 Solution.**

C_B	Basis	b	x_1 (3)	x_2 (-1)	S_1 (0)	S_2 (0)	S_3 (0)	a_1 (-M)	Ratio
0	S_2	3	1	3	0	1	0	0	3/1
0	S_3	4	0	1	0	0	1	0	4/0 = ∞
-M	a_1	2	2	1	-1	0	0	1	2/2 = 1 →
	Z	-2M	-2M-3	-M+1	M	0	0	0	
	S_2	2	0	5/2	1/2	1	0	—	4 →
	S_3	4	0	1	0	0	1	—	4/0 = ∞
	x_1	1	1	1/2	-1/2	0	0	—	-ve
	Z	3	0	5/2	-3/2	0	0	—	
	S_1	4	0	5	1	2	0	—	
	S_3	4	0	1	0	0	1	—	
	x_1	3	1	3	0	1	0	—	
	Z	9	0	10	0	3	0	—	

Since all the index numbers are +ve the solution is optimum.

$\therefore x_1 = 3, \; x_2 = 0$

$Z = 3(3) - 0$

$\therefore Z = 9$

Question Paper Problem, Jan 2003 VTU

Consider the following LPP.

$$\max\ z = x_1 + 2x_2$$

$$\text{STC}\quad 2x_1 + 4x_2 \leq 9$$

$$3x_1 + x_2 \leq 12$$

$$x_1, x_2 \geq 0$$

Solve using simplex method. Does the problem have alternate optimal solutions? If so write down at least two alternate optimal solutions.

$$2x_1 + 4x_2 + S_1 = 9$$

$$3x_1 + x_2 + S_2 = 12$$

$$\max\ z = x_1 + 2x_2 + 0S_1 + 0S_2$$

			1	2	0	0	
			x_1	x_2	S_1	S_2	
0	S_1	9	2	4	1	0	9/4 → min
0	S_2	12	3	1	0	1	12
	z	0	-1	-2	0	0	
	x_2	9/4	1/2	1	1/4	0	9/2
	S_2	39/4	5/2	0	-1/4	1	39/10 → min
	z	9/2	0	0	1/2	0	
	x_2	6/20	0	1	6/20	-1/5	
	x_1	39/10	1	0	-1/10	2/5	
	z	9/2	0	0	1/2	0	

1st soln $x_1 = 0$, $x_2 = 9/4$, max $Z = 9/2$

2nd soln $x_1 = 39/10$, $x_2 = \frac{6}{20}$, max $Z = 9/2$.

Problem of Degeneracy (Tie for minimum Ratio) in LPP:

In the process of improving the solution during simplex procedure, minimum ratio (i.e., X_B/X_K where $X_K > 0$) is determined in the last column of simplex table to fix the key row. But, sometimes this ratio may not be unique, i.e., the key element is not uniquely determined or at the very first iteration, the value of one or more basic variables in the X_B column become equal to zero, this causes the problem of degeneracy.

However, if the minimum ratio is zero for two or more basic variables, degeneracy may result the simplex routine to cycle indefinitely. That is the solution which we have obtained in one iteration may repeat again after few iterations and therefore no optimum solution may be obtained under such circumstances. Fortunately, such phenomenon occurs very rarely in practice.

Method to Resolve Degeneracy (Tie)

Step 1) Pick up the rows for which the minimum non-negative ratio is same (tied). For example first and third row.

Step 2) Rearrange the columns of the usual simplex table so that the columns forming the original unit matrix or (identity matrix) comes first in proper order.

Step 3) Then find the minimum of the ratio

i.e., $\left[\dfrac{\text{elements of first column of identity mtx}}{\text{corresponding elements of key column}}\right]$

only for the rows for which minimum ratio was unique. That is, for the rows first, third etc as picked up in step (1).

(i) If this minimum is attained for third row (say) then this row will determine the key element by intersecting the key column.

(ii) If this minimum is also not unique, then go to next step.

Step 4) Now compute the minimum of the ratio:

$\left[\dfrac{\text{elements of second column of identity matrix}}{\text{corresponding elements of key column}}\right]$

only for the rows for which minimum ratio was not unique in step 3.

(i) If this minimum ratio is unique for the first row (say), then this row will determine the key element by intersecting the key column.

(ii) If this minimum is still not unique then go to next step.

Step 5) Next compute the minimum of the ratio.

$$\left[\frac{\text{elements of third column of identity matrix}}{\text{corresponding elements of key column}}\right]$$

only for the rows for which minimum ratio was not unique in step (4).

(i) If this minimum ratio is unique for the third row (say), then this row will determine the key element by intersecting the key column.

(ii) If this minimum is still not unique, then go on repeating the above procedure till the unique minimum ratio is obtained to resolve the minimum ratio is obtained to resolve the degeneracy. After the resolution of this the, simplex method is applied to obtain the optimum solution.

Problem 1: $\max Z = 2x_1 + x_2$

s.t.c $4x_1 + 3x_2 \leq 12$

$4x_1 + x_2 \leq 8$

$4x_1 - x_2 \leq 8$ $\quad x_1, x_2 \geq 0$

Solution: Transforming the inequality constraints into equations

$$4x_1 + 3x_2 + s_1 = 12$$

$$4x_1 + x_2 + s_2 = 8$$

$$4x_1 - x_2 + s_3 = 8$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0.$$

$$\max Z = 2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3$$

			2	1	0	0	0	
			x_1	x_2	s_1	s_2	s_3	
0	s_1	12	4	3	1	0	0	12/4 = 3
0	s_2	8	4	1	0	1	0	8/4 = 2 } Tie
0	s_3	8	4	-1	0	0	1	8/4 = 2 }
	Z	0	-2	-1	0	0	0	

			s_1	s_2	s_3	x_1 ↓	x_2	s_1/x_1	s_2/x_1
0	s_1	12	1	0	0	4	3		
0	s_2	8	0	1	0	4	1	0/4 } Tie	1/4
0	s_3	8	0	0	1	4	-1	0/4 }	0/4 → min
	Z	0	0	0	0	-2	-1		

S_1 4 1 0 −1 0 | 4 | $4/4 = 1$

S_2 0 0 1 −1 0 | 2 | $0/2 = 0 \rightarrow$ min

x_1 2 0 0 $1/4$ 1 | $-1/4$ | −ve

Z 4 0 0 $1/2$ 0 | $-3/2$

0 0 | $4/1 = 4$ min →

S_1. 4 1 −2 1 0 1 −ve

x_2 0 0 $1/2$ $-1/2$ 1 0 $2 \times 8/1 = 16/1$

x_1 2 0 $1/8$ $1/8$ 0 0

Z 4 0 $3/4$ $-1/4$ 0 0

S_3 4 1 −2 1 0 0

x_2 2 $1/2$ $-1/2$ 0 0 1

x_1 $3/2$ $-1/8$ $3/8$ 0 1 0

Z 5 $1/4$ $1/4$ 0 0 0

Since all the index numbers are +ve the solution is optimal.

$x_1 = 3/2, \; x_2 = 2$

max $Z = 5$

Duality in Linear Programming.

Concept of duality: Every linear programming problem has associated with its another linear programming problem. The original problem is called the "primal" while the other is called its "dual". It is important to note that, in general, either problem can be considered the primal, with the remaining problem its dual. The relationship between the primal and dual problems is actually a very intimate and useful one. The optimal solution of either problem reveals information concerning the optimal solution of the other. If the optimal soln to one is known, then the optimal solution to the other is readily available. This fact is important because the situation can arise where the dual is easier to solve than the primal.

In order to understand the concept of duality let us consider the following problem.

Problem 1:

$$\text{Min } Z = 10x_1 + 16x_2$$

s.t.c

$$8x_1 + 7x_2 \geq 80$$

$$6x_1 + 10x_2 \geq 120$$

and $x_1, x_2 \geq 0$

The above LPP will be considered as the Primal Problem.

Primal Problem	Dual Problem
minimize	maximize
$10x_1 + 16x_2$	$80w_1 + 120w_2$
Subject to the constraints	Subject to the constraints
$8x_1 + 7x_2 \geq 80$	$8w_1 + 6w_2 \leq 10$
$6x_1 + 10x_2 \geq 120$	$7w_1 + 10w_2 \leq 16$
$x_1; x_2 \geq 0$	$w_1, w_2 \geq 0$

Rules for converting primal problem into its Dual

Step 1: First bring the given problem to standard primal form by the following steps.

(a) convert the objective function to maximization form, if not.

(b) If a constraint has inequality sign $\geq$, then multiply both sides by -1 and make the inequality sign $\leq$.

(c). If a constraint has an equality sign ($=$), then it is replaced by two constraints involving the inequalities as given below.

For example: $3x_1 + 2x_2 = 4$ is replaced by two opposite inequalities ($\leq$ & $\geq$) constraints

$3x_1 + 2x_2 \leq 4$ & $3x_1 + 2x_2 \geq 4$.

The second inequality with $\geq$ sign, can be further written as $-3x_1 - 2x_2 \leq -4$.

(d) Every unrestricted variable is replaced by the difference of two non-negative variables

[Note: The dual variables that correspond to primal equality constraints must be unrestricted in sign, and those associated with the primal inequalities must be non-negative].

Step II obtain the standard primal form of the given LPP in which.

(i) all the constraints have $\leq$ sign; the objective function is of maximization form or

(ii) all the constraints have $\geq$ sign; where the objective function is of minimization form

Step III. Then the dual of the given problem is obtained by:

(i) minimizing the objective function instead of maximizing it,

(ii) Transposing the right side constants of the constraints to the coefficients of the objective function.

(iii) Transposing the columns of constraint coefficients to rows.

(iv) changing the inequalities from $\leq$ to $\geq$ sign and,

(v) transposing the coefficients of the objective function to the right side constants.

Problem 2: Find the dual of the following primal problem:

Minimize $Z = 2x_2 + 8x_3$

Stc $3x_1 + x_2 \geq 12$

$2x_1 + x_2 + 6x_3 \leq 6$

$5x_1 - x_2 + 3x_3 = 4$

$x_1, x_2, x_3 \geq 0$.

Max $Z' = -2x_2 - 8x_3$ where $Z' = -Z$

$3x_1 + x_2 \geq 12 \Rightarrow -3x_1 - x_2 \leq -12$

Eqn. $5x_1 - x_2 + 3x_3 = 4$

$5x_1 - x_2 + 3x_3 \leq 4$

$5x_1 - x_2 + 3x_3 \geq 4 \Rightarrow -5x_1 + x_2 - 3x_3 \leq -4$

original problem in the standard primal from is

max $Z' = 0x_1 - 2x_2 - 8x_3$

Stc $-3x_1 - x_2 \leq -12$

$2x_1 + x_2 + 6x_3 \leq 6$

$5x_1 - x_2 + 3x_3 \leq 4$

$-5x_1 + x_2 - 3x_3 \leq -4$

$x_1, x_2, x_3 \geq 0$

By applying the dual rules.

min $Z' = -12w_1 + 6w_2 + 4w_3 - 4w_4$

Stc. $-3w_1 + 2w_2 + 5w_3 - 5w_4 \geq 0$

$-w_1 + w_2 - w_3 + w_4 \geq -2$

$6w_2 + 3w_3 - 3w_4 \geq -8$

$w_1, w_2, w_3, w_4 \geq 0.$

Problem 2: (Feb 97)

Give the dual of the LPP

$$\min Z = 2x_1 + 3x_2 + 4x_3$$

s.t.c $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 = 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2 \geq 0$ and x_3 is unrestricted.

Soln: Since x_3 is unrestricted

$x_3 = x_3' - x_3''$;

where $x_3' \geq 0$ $x_3'' \geq 0$

∴ Standard primal is

$\max Z' = -2x_1 - 3x_2 - 4(x_3' - x_3'')$; $Z' = -Z$,

s.t.c $-2x_1 - 3x_2 - 5(x_3' - x_3'') \leq -2$

$3x_1 + x_2 + 7(x_3' - x_3'') \leq 3$

$-3x_1 - x_2 - 7(x_3' - x_3'') \leq -3$

$x_1 + 4x_2 + 6(x_3' - x_3'') \leq 5$

$x_1, x_2, x_3', x_3'' \geq 0$

∴ dual fo the given primal is

$\min Z_w' = -2w_1 + 3(w_2' - w_2'') + 5w_3$

s.t.c. $-2w_1 + 3(w_2' - w_2'') + w_3 \geq -2$

$-3w_1 + (w_2' - w_2'') + 4w_3 \geq -3$

$-5w_1 + 7(w_2' - w_2'') + 6w_3 \geq -4$

$5w_1 - 7(w_2' - w_2'') - 6w_3 \geq 4$

$w_1, w_2', w_2'', w_3 \geq 0$

OR.

$\min z'w = -2w_1 + 3w_2 + 5w_3$

s.t. $-2w_1 + 3w_2 + w_3 \geq -2$

$-3w_1 + w_2 + 4w_3 \geq -3$

$5w_1 - 7w_2 - 6w_3 = 4$

$w_1, w_3 \geq 0$ w_2 is unrestricted.

Comparison of Solutions to the primal & its dual:

Primal Problem.

$\max Z = 40x_1 + 50x_2$

s.t. $2x_1 + 3x_2 \leq 3$

$8x_1 + 4x_2 \leq 5$

$x_1, x_2 \geq 0.$

writing the problem in standard Simplex form.

$\max Z = 40x_1 + 50x_2$

s.t. $2x_1 + 3x_2 + S_1 = 3$

$8x_1 + 4x_2 + S_2 = 5$

$x_1, x_2, S_1, S_2 \geq 0$

$\max Z = 40x_1 + 50x_2 + 0S_1 + 0S_2$

			40	50	0	0	
			x_1	x_2	S_1	S_2	
0	S_1	3	2	3	1	0	$3/3 = 1$ min →
0	S_2	5	8	4	0	1	$5/4 = 1.25$
	Z	0	−40	−50	0	0	

x_2	1	2/3	1	1/3	0	$3/2 = 1.5$
S_2	1	16/3	0	−4/3	1	$3/16 = 0.1875$ → min
Z	50	−20/3	0	50/3	0	

x_2	7/8	0	1	1/2	−1/8
x_1	3/16	1	0	−1/4	3/16
Z	205/4	0	0	15	5/4

Dual problem

$$\min Z_w = 3w_1 + 5w_2$$

s.t.c $2w_1 + 8w_2 \geq 40$

$3w_1 + 4w_2 \geq 50$

$w_1, w_2 \geq 0$

writing the problem in standard simplex form where $Z'_w = -Z_w$

$$\max Z'_w = -3w_1 - 5w_2$$

s.t.c $2w_1 + 8w_2 - S_1 + a_1 = 40$

$3w_1 + 4w_2 - S_2 + a_2 = 50$

$w_1, w_2, S_1, S_2, a_1, a_2 \geq 0$

$$\max Z'_w = -3w_1 - 5w_2 + 0S_1 + 0S_2 - Ma_1 - Ma_2$$

			-3	-5	0	0	-M	-M	
			w_1	w_2	S_1	S_2	a_1	a_2	
-M	a_1	40	2	8	-1	0	1	0	$40/8 = 5$ →
-M	a_2	50	3	4	0	-1	0	1	$50/4 = 12.5$
	Z_w	-90M	-5M+3	-12M+5	M	M	0	0	
	w_2	5	1/4	1	-1/8	0	1/8	0	20
	a_2	30	2	0	1/2	-1	-1/2	1	15 → min
	Z_w	-30M-25	$\frac{-8M+7}{4}$	0	$\frac{-4M+5}{8}$	M	$\frac{12M-5}{8}$	0	
	w_2	(5/4)	0	1	-3/16	1/8	3/16	-1/8	
	w_1	(15)	1	0	1/4	-1/2	-1/4	1/2	
	Z_w	-205/4	0	0	(3/16)	(7/8)	$\frac{16M-3}{16}$	$\frac{8M-7}{8}$	

from the above comparison, Soln to a primal can provide a soln to its dual

Assignment:

Apply the principle of duality to solve the LP problem.

$\max Z = 3x_1 - 2x_2$

stc $x_1 + x_2 \leq 5$

$x_1 \leq 4$

$1 \leq x_2 \leq 6$ and $x_1, x_2 \geq 0$

Soln: writing the given problem in std primal form

$\max Z = 3x_1 - 2x_2$

stc. $x_1 + x_2 \leq 5$

$x_1 + 0x_2 \leq 4$

$0x_1 + x_2 \leq 6$

$0x_1 - x_2 \leq -1$ and $x_1, x_2 \geq 0$.

Obtain the dual of the following LP problems

(3) $\min Z = 4x_1 + 6x_2 + 3x_3$

S.t.c $3x_1 + 4x_2 + x_3 \geq 10$

$-2x_1 - 3x_2 + 2x_3 \leq -5$

$x_1 - 2x_2 - 3x_3 \leq -1$

$3x_1 + 2x_2 + 2x_3 \geq 5$

$x_1, x_2, x_3 \geq 0.$

Answer: $\max Z = 10w_1 + 5w_2 + w_3 + 5w_4$

S.t.c $3w_1 + 2w_2 - w_3 + 3w_4 \leq 4$

$4w_1 + 3w_2 + 2w_3 + 2w_4 \leq 6$

$w_1 - 2w_2 + 3w_3 + 2w_4 \leq 3$

$w_1, w_2, w_3, w_4 \geq 0.$

Steps for Dual Simplex Method

Step 1) Convert the minimization LPP into maximization, If it is in the minimization form.

Step 2) Convert "$\geq$" type inequalities of given LPP into those of "$\leq$" type by multiplying the corresponding constraints by -1.

Step 3) Introduce slack variables in the constraints of the given problem and obtain an initial basic solution. Put this solution in the starting dual simplex table.

Step 4) Test the nature of net evaluation $(z_j - c_j)$ in the starting table.

(i) If all net evaluation $(Z_j - c_j)$ & x_{Bi} are non-negative for all i & j, then an optimum basic feasible solution has been attained.

(ii) If all net evaluations $(Z_j - c_j)$ are non-negative and at least one basic variable x_{Bi} is negative, then go to Step 5.

(iii) If at least one net evaluation $(Z_j - c_j)$ is -ve, the method is not applicable to the given problem.

Step 5) Select the most -ve x_{Bi} which becomes the key row.

Step 6) Test the nature of x_{rj}, $j = 1, 2, \ldots n$.

(i) If all x_{rj} are non-negative, then feasible solution does not exists for the given problem.

(ii) If at least one x_{rj} is -ve, compute the replacement ratios

$$\left(\frac{Z_j - C_j}{x_{rj}}, \; x_{rj} < 0 \right), \quad j = 1, 2, \ldots n.$$

and choose the maximum of these which will become the key column.

Step 7) Test the new iterated dual simplex table for optimality, if not repeat the above steps.

Problem 1: Solve the following problem by dual simplex method.

$$\max Z = -2x_1 - x_2$$

s.t.c
$$3x_1 + x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \geq 3$$
$$x_1, x_2 \geq 0.$$

Soln: $\max Z = -2x_1 - x_2$

s.t.c
$$-3x_1 - x_2 \leq -3$$
$$-4x_1 - 3x_2 \leq -6$$
$$-x_1 - 2x_2 \leq -3$$
$$x_1, x_2 \geq 0$$

$$\therefore -3x_1 - x_2 + S_1 = -3$$
$$-4x_1 - 3x_2 + S_2 = -6$$
$$-x_1 - 2x_2 + S_3 = -3$$

and $\max Z' = -2x_1 - x_2 + 0S_1 + 0S_2 + 0S_3$

			-2	-1	0	0	0
			x_1	x_2	S_1	S_2	S_3
0	S_1	-3	-3	-1	1	0	0
0	S_2	-6	-4	-3	0	1	0
0	S_3	-3	-1	-2	0	0	1
	Z	0	2	1	0	0	0
	S_1	-1	-5/3	0	1	-1/3	0
	x_2	2	4/3	1	0	-1/3	0
	S_3	1	5/3	0	0	-2/3	1
	Z	-2	2/3	0	0	1/3	0
	x_1	3/5	1	0	-3/5	1/5	0
	x_2	6/5	0	1	4/5	-3/5	0
	S_3	0	0	0	1	-1	1
	Z	-12/5	0	0	2/5	1/5	0

* Note: In the constant column -6 is most negative. ∴ select that as the key row. Then select the key column among:

$\max\left[\frac{2}{-4}, \frac{1}{-3}\right] \Rightarrow \max[-0.5, -0.33]$

∴ max is $-1/3$

$\max\left[\frac{2/3}{-5/3}, \frac{1/3}{-1/3}\right]$

$\max[-0.4, -1]$ ✓

Since all the elements in the constant column are +ve the solution is feasible at this stage. And the optimal solution is

$x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$, max $Z = -\frac{12}{5}$.

<u>Problem 2</u>: using dual simplex algorithm solve the following LPP.

Min $Z = 5x_1 + 6x_2$

Stc $x_1 + 3x_2 \geq 76$

$6x_1 + 7x_2 \leq 198$ $\quad x_1, x_2 \geq 0$.

Problem 4: Solve the following LPP either by Big M method or by dual simplex method.

$$\text{Min } Z = 2x_1 + x_2$$

s.t.c

$$x_1 + x_2 = 4$$

$$2x_1 - x_2 \geq 3$$

$$x_1, x_2 \geq 0.$$

[Feb 2002, Jan 2005 Common Paper]