

TRANSPORTATION PROBLEM

Introduction

A special class of linear programming problem is **Transportation Problem**, where the objective is to minimize the cost of distributing a product from a number of **sources** (e.g. factories) to a number of **destinations** (e.g. warehouses) while satisfying both the supply limits and the demand requirement. Because of the special structure of the Transportation Problem the Simplex Method of solving is unsuitable for the Transportation Problem. The model assumes that the distributing cost on a given route is directly proportional to the number of units distributed on that route. Generally, the transportation model can be extended to areas other than the direct transportation of a commodity, including among others, inventory control, employment scheduling, and personnel assignment.

The transportation problem special feature is illustrated here with the help of following Example.

Example:

Suppose a manufacturing company owns three factories (sources) and distribute his products to five different retail agencies (destinations). The following table shows the capacities of the three factories, the quantity of products required by the various retail agencies and the cost of shipping one unit of the product from each of the three factories to each of the five retail agencies.

Factories	Retail Agency					Capacity
	1	2	3	4	5	
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Usually the above table is referred as Transportation Table, which provides the basic information regarding the transportation problem. The quantities inside the table are known as transportation cost per unit of product. The capacity of the factories 1, 2, 3 is 50, 100 and 150 respectively. The requirement of the retail agency 1, 2, 3, 4, 5 is 100,60,50,50, and 40 respectively.

In this case, the transportation cost of one unit

- from factory 1 to retail agency 1 is 1,
- from factory 1 to retail agency 2 is 9,
- from factory 1 to retail agency 3 is 13, and so on.

A transportation problem can be formulated as linear programming problem using variables with two subscripts.

Let

x_{11} =Amount to be transported from factory 1 to retail agency 1

x_{12} = Amount to be transported from factory 1 to retail agency 2

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.....

.....

.....

x_{35} = Amount to be transported from factory 3 to retail agency 5.

Let the transportation cost per unit be represented by $C_{11}, C_{12}, \dots, C_{35}$ that is $C_{11}=1, C_{12}=9$, and so on.

Let the capacities of the three factories be represented by $a_1=50, a_2=100, a_3=150$.

Let the requirement of the retail agencies are $b_1=100, b_2=60, b_3=50, b_4=50$, and $b_5=40$. Thus, the problem can be formulated as

Minimize

$$C_{11}x_{11}+C_{12}x_{12}+\dots+C_{35}x_{35}$$

Subject to:

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = a_1$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = a_2$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = a_3$$

$$x_{11} + x_{21} + x_{31} = b_1$$

$$x_{12} + x_{22} + x_{32} = b_2$$

$$x_{13} + x_{23} + x_{33} = b_3$$

$$x_{14} + x_{24} + x_{34} = b_4$$

$$x_{15} + x_{25} + x_{35} = b_5$$

$$x_{11}, x_{12}, \dots, x_{35} \geq 0.$$

Thus, the problem has 8 constraints and 15 variables. So, it is not possible to solve such a problem using simplex method. This is the reason for the need of special computational procedure to solve transportation problem. There are varieties of procedures, which are described in the next section.

Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

Step 1: Determine a starting basic feasible solution, using any one of the following three methods

North West Corner Method

Least Cost Method

Vogel Approximation Method

Step 2: Determine the optimal solution using the following method

1. MODI (Modified Distribution Method) or UV Method.

Basic Feasible Solution of a Transportation Problem

The computation of an initial feasible solution is illustrated in this section with the help of the example discussed in the previous section. The problem in the example 1.1 has 8 constraints and 15 variables we can eliminate one of the constraints since $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 + b_4 + b_5$. Thus now the problem contains 7 constraints and 15 variables. Note that any initial (basic) feasible solution has at most 7 non-zero X_{ij} . Generally, any basic feasible solution with m sources (such as factories) and n destination (such as retail agency) has at most $m + n - 1$ non-zero X_{ij} .

The special structure of the transportation problem allows securing a non artificial basic feasible solution using one the following three methods.

North West Corner Method

Least Cost Method

Vogel Approximation Method

The difference among these three methods is the **quality** of the initial basic feasible solution they produce, in the sense that a better that a better initial solution yields a smaller objective value. Generally the Vogel Approximation Method produces the **best** initial basic feasible solution, and the North West Corner Method produces the **worst**, but the North West Corner Method involves least computations.

North West Corner Method:

The method starts at the North West (upper left) corner cell of the tableau (variable x_{11}).

Step -1: Allocate as much as possible to the selected cell, and adjust the associated amounts of capacity (supply) and requirement (demand) by subtracting the allocated amount.

Step -2: Cross out the row (column) with zero supply or demand to indicate that no further assignments can be made in that row (column). If both the row and column becomes zero simultaneously, cross out one of them only, and leave a zero supply or demand in the uncrossed out row (column).

Step -3: If exactly one row (column) is left uncrossed out, then stop. Otherwise, move to the cell to the right if a column has just been crossed or the one below if a row has been crossed out. Go to step -1.

Example:

Consider the problem discussed in Example to illustrate the North West Corner Method of determining basic feasible solution.

	Retail Agency					
Factories	1	2	3	4	5	Capacity
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

The allocation is shown in the following tableau:

								Capacity			
	1	9	13		36		51				
	50							50			
	24	12	16		20		1				
	50	50						100	50		
	14	33	1		23		26				
		10	50	50			40				
								150-140	90	40	
Requirement	100	60	50	50			40				
	50	10									

The starting basic solution is given as

$$x_{11} = 50,$$

$$x_{21} = 50, x_{22} = 50$$

$$x_{32} = 10, x_{33} = 50, x_{34} = 50, x_{35} = 40$$

The corresponding transportation cost is

$$50 * 1 + 50 * 24 + 50 * 12 + 10 * 33 + 50 * 1 + 50 * 23 + 40 * 26 = 4420$$

It is clear that as soon as a value of X_{ij} is determined, a row (column) is eliminated from further consideration. The last value of X_{ij} eliminates both a row and column. Hence a feasible solution computed by North West Corner Method can have at most $m + n - 1$ positive X_{ij} if the transportation problem has m sources and n destinations.

Least Cost Method

The least cost method is also known as matrix minimum method in the sense we look for the row and the column corresponding to which C_{ij} is minimum. This method finds a better initial basic feasible solution by concentrating on the cheapest routes. Instead of starting the allocation with the northwest cell as in the North West Corner Method, we start by allocating as much as possible to the cell with the smallest unit cost. If there are two or more minimum costs then we should select the row and the column corresponding to the lower numbered row. If they appear in the same row we should select the lower numbered column. We then cross out the satisfied row or column, and adjust the amounts of capacity and requirement accordingly. If both a row and a column is satisfied simultaneously, only one is crossed out. Next, we look for the uncrossed-out cell with the smallest unit cost and repeat the process until we are left at the end with exactly one uncrossed-out row or column.

Example:

The least cost method of determining initial basic feasible solution is illustrated with the help of problem presented in the section 1.1.

					Capacity
	1	9	13	36	51
50					50
24		12	16	20	1
	60				40
14		33	1	23	26
50			50	50	

					150 100 50
Requirement	100	60	50	50	40
	50				

The Least Cost method is applied in the following manner:

We observe that $C_{11}=1$ is the minimum unit cost in the table. Hence $X_{11}=50$ and the first row is crossed out since the row has no more capacity. Then the minimum unit cost in the uncrossed-out row and column is $C_{25}=1$, hence $X_{25}=40$ and the fifth column is crossed out. Next $C_{33}=1$ is the minimum unit cost, hence $X_{33}=50$ and the third column is crossed out. Next $C_{22}=12$ is the minimum unit cost, hence $X_{22}=60$ and the second column is crossed out. Next we look for the uncrossed-out row and column now $C_{31}=14$ is the minimum unit cost, hence $X_{31}=50$ and crossed out the first column since it was satisfied. Finally $C_{34}=23$ is the minimum unit cost, hence $X_{34}=50$ and the fourth column is crossed out.

So that the basic feasible solution developed by the Least Cost Method has transportation cost is

$$1 * 50 + 12 * 60 + 1 * 40 + 14 * 50 + 1 * 50 + 23 * 50 = 2710$$

Note that the minimum transportation cost obtained by the least cost method is much lower than the corresponding cost of the solution developed by using the north-west corner method.

Vogel Approximation Method (VAM):

VAM is an improved version of the least cost method that generally produces better solutions. The steps involved in this method are:

Step 1: For each row (column) with strictly positive capacity (requirement), determine a **penalty** by subtracting the **smallest** unit cost element in the row (column) from the next **smallest** unit cost element in the same row (column).

Step 2: Identify the row or column with the **largest penalty** among all the rows and columns. If the penalties corresponding to two or more rows or columns are equal we select the topmost row and the extreme left column.

Step 3: We select X_{ij} as a basic variable if C_{ij} is the **minimum cost** in the row or column with **largest penalty**. We choose the numerical value of X_{ij} as high as possible subject to the row and the column constraints. Depending upon whether a_i or b_j is the smaller of the two i^{th} row or j^{th} column is crossed out

Step 4: The Step 2 is now performed on the uncrossed-out rows and columns until all the basic variables have been satisfied.

Example 1.4:

Consider the following transportation problem

Origin	Destination				a_i
	1	2	3	4	
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
b_j	60	40	30	110	240

Note: a_i =capacity (supply)

b_j =requirement (demand)

Now, compute the penalty for various rows and columns which is shown in the following table:

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22	17	4	120	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	
Row Penalty	4	15	8	3		

Look for the highest penalty in the row or column, the highest penalty occurs in the **second column** and the minimum unit cost i.e. c_{ij} in this column is $c_{12}=22$. Hence assign 40 to this cell i.e. $x_{12}=40$ and cross out the second column (since second column was satisfied. This is shown in the following table:

Origin	Destination				a _i	Column Penalty
	1	2	3	4		
1	20	22	40	17	4	80 13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b _j	60	40	30	110	240	
Row Penalty	4	15	8	3		

The next highest penalty in the uncrossed-out rows and columns is 13 which occur in the **first row** and the minimum unit cost in this row is $c_{14}=4$, hence $x_{14}=80$ and cross out the first row. The modified table is as follows:

Origin	Destination				a _i	Column Penalty
	1	2	3	4		
1	20	22	17		40	80 13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b _j	60	40	30	110	240	
Row Penalty	4	15	8	3		

The next highest penalty in the uncrossed-out rows and columns is 11 which occurs in the **third column** and the **minimum cost** in this column is $c_{23}=9$, hence $x_{23}=30$ and cross out the third column with adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination				a _i	Column Penalty
	1	2	3	4		
1	20	22	17	4	0	13
		40		80		
2	24	37	9	7	40	17
			30			
3	32	37	20	15	50	17
b _j	60	40	30	110	240	
Row Penalty	8	15	11	8		

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the **second row** and the **smallest cost** in this row is $c_{24} = 7$ hence $x_{24} = 30$ and cross out the fourth column with the adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination				a _i	Column Penalty
	1	2	3	4		
1	20	22	17	4	0	13
		40		80		
2	24	37	9	7	10	17
			30	30		
3	32	37	20	15	50	17
b _j	60	40	30	110	240	
Row Penalty	8	15	11	8		

The left out demand that is unfulfilled is Destination 1 and the remaining supply of 10 from supply 2 and 50 from supply 3 is allocated for cells C_{21} and C_{31} respectively. The final table is as follows.

Origin	Destination				a _i	Column
	1	2	3	4		Penalty
1	20	22 40	17	4	0	13
2	24 10	37	9 30	7	0	17
3	32 50	37	20	15	0	17
b _j	60	40	30	110	240	
Row Penalty	8	15	8	8		

The transportation cost corresponding to this choice of basic variables is

$$22 * 40 + 4 * 80 + 9 * 30 + 7 * 30 + 24 * 10 + 32 * 50 = 3520$$

Unbalanced Transportation Problem

In the previous section we discussed about the balanced transportation problem i.e. the total supply(capacity) at the origins is equal to the total demand (requirement) at the destination. In this section we are going to discuss about the unbalanced transportation problems i.e. when the total supply is not equal to the total demand, which are called as **unbalanced transportation problem**.

In the unbalanced transportation problem if the total supply is more than the total demand then we introduce an additional column which will indicate the surplus supply with transportation cost zero. Similarly, if the total demand is more than the total supply an additional row is introduced in the transportation table which indicates unsatisfied demand with zero transportation cost.

Example:

Consider the following unbalanced transportation problem

		Warehouses			
Plant		w ₁	w ₂	w ₃	Supply
X		20	17	25	400
Y		10	10	20	500
Demand		400	400	500	

In this problem the demand is 1300 whereas the total supply is 900. Thus, we now introduce an additional row with zero transportation cost denoting the unsatisfied demand. So that the modified transportation problem table is as follows:

		Warehouses			
Plant		w ₁	w ₂	w ₃	Supply
X		20	17	25	400
Y		10	10	20	500
Unsatisfied		0	0	0	
Demand		400	400	500	1300

Now we can solve as balanced problem discussed as in the previous sections.