OPERATION RESEARCH

UNIT-I Introduction

ORIGIN and DEVELOPMENT OF OR

Operation Research may be described as a scientific approach to decision making.

The term operations research was first coined in 1940 by Mc Closky and Trefthen in Bowdsey of the United Kingdom. This new science came into existence during the second world war of 1939-45. At that time, the military management in England called upon a team of scientists to study the strategic and tactical Problems related to their limited military resources. It was thus necessary to decide upon the most effective utilization of their military resources such as ocean transport, effective bombing etc.

Since, the military team was dealing with research on military operations, the name operations research was given to it.

Following the end of the war, the success of the military teams attracted the attention of industrial managers who were seeking solutions to their Complex executive-type problems.

With a view to increase the impact of OR, the Operations Research society of America (ORSA) was formed in 1950.

In 1957, the international Federation of OR Societies was established.

In India Operation Research came into existence in 1949 with the opening of an OR unit at Regional Research Laboratory, Hyderabad. At the same time another group was set up in defence science laboratory which was dedicated to the problems of stores, purchase and planning.

In 1953, an OR unit was established in Indian Statistical Institute, Calcutta for the application of OR methods in national planning and survey.

OR society of India (ORSI) was formed in 1955 and became a member of the International Federation of OR societies in 1959.

Definition of OR

It is very difficult to define OR, mainly because of the fact that its boundaries are not clearly marked. It has been defined so far in various ways and it is changing according to the development of the subject as follows:

① OR is a scientific method of providing executive departments with a quantitative basis for making decision regarding the operations under their control.

P.M. Morse & G.E. Kimbal [1946]

② OR is defined as the attack of modern science on complex problems arising in the direction and management of large systems of men, machines, materials and money on industry, business, government and defence. The distinctive approach is to develop a scientific model of the system, incorporating measurement of factors such as chance and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management determine its policy and actions scientifically.

- Council of UK operational Research society (1971)

Reasons for unsatisfactory definitions of OR.

① OR is not a science like any well-defined physical or biological phenomena. OR is essentially a collection of mathematical techniques and tools which in association with a system approach are applied to solve practical economic or engineering problems. Thus it is very difficult to define OR precisely.

Scope of Operations Research (Application Areas)

Operations research has been used successfully in many areas of research for military, government and industry. There is a great scope for economists, statisticians, administrators, technicians and even politicians to solve their problem by an OR approach. Besides this, OR is also useful in the following fields.

1. Agriculture:
   (i) For optimum allocation of land in various crops in accordance with the climatic conditions.
   (ii) optimum distribution of water from various resources.

2. Industry:
   (i) To decide optimum allocation of various resources such as men, machines, materials, money, time etc.

3. Finance: for carefull planning of finance, OR helps to
   (i) maximize the per capita income with minimum resources.
   (ii) Find a profit plan and the best replacement policies etc.

4) Marketing : (a) Market research and sales forecasting
(i) To distribute the products for sale such that the total cost of transportation is minimum.
(ii) The size of the stock to meet the future demand.
(iii) Selection of best advertising media with respect to time, cost ~~and~~ effectiveness

5) Personnel management :
(i) To appoint most suitable persons on minimum salary.
(ii) To determine the number of persons to be appointed on full time basis when the work is seasonal.

6) Production management :
(i) To find the quantity of items to be produced.
(ii) Planning, scheduling and sequencing the production run by proper allocation of machines.
(iii) To determine the optimum product mix.
(iv) Selection, location and design of production plants.

7) Purchasing and Procurement :
(i) Determination of quantities and time of purchases
(ii) Rules for buying or supplying and (Plan) ~~stable~~ for varying prices.
(iii) Vendor development and bidding policies.

(iv) Inventory control and replacement policies.

(v) Strategies for exploration and exploitation of material resources.

8) ~~Production~~ Industry management

(a) Physical Distribution - Location, size of distribution centres retail outlets and distribution policies.

(b) Facilities planning - Number and location of factories, service centres etc. Determining the loading and unloading facilities and transportation schedules.

(c) Manufacturing - Production planning and Job sequencing. Employment training, lay offs and optimum product mix. Inspection and quality control.

(d) maintenance and project scheduling - maintenance policies, optimum maintenance crew size, Project scheduling and allocation of resources.

Main Phases of operations Research.

The procedure for an OR study involves the following major phases:

① Definition of the Problem.
② Construction of the model.
③ Solution of the model.
④ Validation of the model.
⑤ Implementation of the solution.

Phase 3, dealing with model solution, is the best defined and generally the easiest to implement in an OR study, because it deals mostly with precise mathematical models. Implementation of the remaining phases is more an art than a theory.

1. Problem definition involves defining the scope of the problem under investigation. This requires the problem to be formulated in the form of an appropriate model. The aim is to identify three principal elements of the decision problem:
(i) Description of the decision alternatives.
(2) Determination of the objective of the study.
(3) specification of the limitations under which the modeled system operates.

2. Model Construction entails an attempt to translate the problem definition into mathematical relationships

A mathematical model should include decision variables, objective function, constraints or restrictions etc., The advantage of a mathematical model is that it describes the problem more concisely which makes the overall structure of the problem more comprehensible, and it also helps to reveal important cause-and-effect relationships.

3. model solution is devoted to the computation of those values of decision variables which maximizes or minimizes the objective function depending on their nature. It is important in operations research to determine an optimal or best solution for the problem.

An important aspect of the model solution phase is sensitivity analysis. It deals with obtaining additional information about the behavior of the optimum solution when the model undergoes some parameter changes.

Sensitivity Analysis is particularly needed when the parameters of the model cannot be estimated accurately. In these cases, it is important to study the behavior of the optimum solution in the neighborhood of the estimated parameters.

4. Model validity: checks whether or not the proposed model does what it purports to do - that is, does it predict adequately the behavior of the system under study? (No surprises).

On the formal side, a common method for checking the validity of a model is to compare its output with historical output data. The model is valid if, under similar input conditions, it reasonably duplicates past performance.

If the proposed model represents a new (nonexisting) system, no historical data would be available. In such cases, we may use simulation as an independent tool for verifying the output of the mathematical model.

5. Implementation of the solution of a validated model involves the translation of the results into understandable operating instructions to be issued to the people who will administer the recommended system. The burden of this task lies primarily with the OR team.

OR models and solution generation

In OR, we do not have a single general technique to solve all mathematical models that can arise in practice. Instead, the type and complexity of the mathematical model dictate the nature of the solution method.

The most prominent OR technique is linear programming. It is designed for models with linear objective and constraint functions. Other techniques include integer programming (in which the variables assume integer values), dynamic programming (in which the original model can be decomposed into more manageable subproblems), network programming (in which the problem can be modeled as a network), and nonlinear programming (in which functions of the model are nonlinear). These are only a few among many available OR tools.

A peculiarity of most OR techniques is that solutions are not generally obtained in (formulalike) closed forms. Instead, they are determined by algorithms. An algorithm provides fixed computational rules that are applied repetitively to the problem, with each repetition (called iteration) moving the solution closer to the optimum. Because the computations associated with each iteration are typically tedious and voluminous, it is imperative that these algorithms be executed on the computer.

Some mathematical models may be so complex that it is impossible to solve them by any of the available optimisation algorithm. In such cases, it may be necessary to abandon the search for the optimal solution and simply seek a good solution using heuristics or rules of thumb.

Advantages of Operations Research

(1) Optimum use of production factors: Linear programming techniques indicate how a manager can most effectively employ his production factors by more efficiently selecting and distributing these elements.

(2) Improved quality of decision: The computation table. gives a clear picture of the happenings within the basic restrictions and the possibilities of compound behaviour of the elements involved in the problem. The effect on the profitability due to changes in the production pattern would be clearly indicated in the solution.

(3) Preparation of future managers: OR techniques substitute a means for improving the knowledge and skill of young managers.

(4). Modification of mathematical solution: OR presents a possible practical solution when one exists, but it is the responsibility of the manager to accept or modify the solution before its use. The effect of these modifications may be evaluated from the computational steps and tables.

(5) Alternative solutions: OR techniques will suggest all the alternative solutions available for the same problem so that the management may decide on the basis of its strategies.

Disadvantages of Operations Research

① Practical application : Formulation of an industrial problem to an O.R. set programme is a difficult task.

② Reliability of the proposed solution : A non-linear relationship is changed to linear for fitting the problem to linear programming pattern. This may disturb the solution.

③ Magnitude of computation : O.R. tries to find out the optimal solution by taking all the factors into account. In the practical problems these factors are numerous and expressing them in quantity and establishing relationships among these requires huge calculations and also costilier. Thus the use of O.R. is limited to very large organizations.

④ Absence of Quantification : O.R. provides solution only when all the elements related to a problem are quantified. The tangible factors such as price, product, etc can be expressed in terms of quantity, but intangible factors such as human relations, etc cannot be quantified. Thus, these intangible elements of the problem are excluded from the study, though they may be equal or more important than quantifiable elements.

(5) Distance between managers and operation research: O.R. being specialists job, requires mathematician or statistician who might not be aware of the business problems. Similarly, a manager fails to understand the complex working of O.R.

LINEAR PROGRAMMING

Introduction:

Many management decisions are concerned with the problem of planning activity. In each case, these will be limited resources and it becomes necessary to utilize these resources so as to yield maximum production or to minimize the cost of production, or to give maximum profits etc. Such problems are referred to as the problems of constrained optimization. Linear programming is a mathematical modeling technique designed to optimize the usage of limited resources. The term 'Linear' indicates that all the relationships involved in a particular problem are linear. While, the term 'Programming' refers to the process of determining a particular program or plan of action among the several alternative courses of action available.

A linear form means a mathematical expression of the type,

$A_1x_1 + A_2x_2 + \ldots + A_nx_n$, where

$A_1, A_2, A_3 \ldots A_n$ are constants and

$x_1, x_2, x_3 \ldots x_n$ are variables.

The solution for a linear programming problem (LPP) can be obtained by a set of simultaneous equations.

However, a unique solution for a set of simultaneous equations with n-variables $x_1, x_2 \ldots x_n$ in which atleast one of the variables is non-zero, can be obtained if there are exactly n relations.

On the otherhand, if the number of relations are greater than or less than n-variables, a unique solution does not exist, but a number of solutions will result.

In practical problems, the number of relations are not equal to the number of variables and many of the relations are in the form of inequalities ($\leq$ or $\geq$) such ~~that~~ kind of problems are known as Linear Programming Problems (LPP).

Definition of Linear Programming Problem

Linear programming problems deals with the optimization (maximization/minimization) of a linear function of variables called the 'objective function' subjected to a set of linear equations and/or inequalities called the 'constraints' or 'restrictions'.

mathematical formulation of LPP.

Example 1

A company manufactures two types of products P_1 and P_2 and sells them at a profit of Rs. 3/- on product P_1 and Rs 4/- on product P_2. Each product is processed on two machines M_1 and M_2. Product P_1 requires 2 minutes of processing time on M_1 and 3 minutes on machine M_2; while product P_2 requires one minute on M_1 and 2 minutes on M_2. The machine M_1 is available for not more than 7 hours 50 minutes, while machine M_2 is available for 9 hours during any working day. Formulate this as an LPP.

Solution :

Let x_1 be the number of P_1 products and x_2 be the number of P_2 products. Analyze the problem and arrange the given information as shown in the table 1.0

Table. 1.0

	Products		
	P_1 (x_1 number)	P_2 (x_2 number)	
Profit (Rs)	3	4	
Processing time on machine M_1 (min)	2	1	Availability 7 hours 50 mins
Processing time on machine M_2 (min)	3	2	Availability 9 hours

The profit on product P_1 is Rs 3/-, therefore $3x_1$ will be the profit on selling x_1 number of P_1 products. Similarly $4x_2$ will be the profit on selling x_2 units of P_2 products. Thus, the total profit on selling x_1 number of P_1 products and x_2 number of P_2 products is given by.

$$Z = 3x_1 + 4x_2$$

A logical objective for the company is to increase the profit as much as possible. Therefore the objective function of the problem is to

maximize $Z = 3x_1 + 4x_2$

The machine M_1 takes 2 minutes of processing time for product P_1 and 1 minute of processing time for product P_2. Therefore, the total processing time required on machine M_1 is given by:

$$2x_1 + 1x_2$$

But the availability of machine M_1 is only 7 hours and 50 minutes or 470 minutes. Therefore, the first constraint is

$$2x_1 + x_2 \leq 470$$

Similarly, the processing time on the machine M_2 and the corresponding second constraint is

$$3x_1 + 2x_2 \leq 540$$

Since it is not possible to produce negative number of products, the non-negativity restrictions are:

$$x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0$$

Therefore the LPP of the given problem is to

$$\boxed{\begin{aligned} &\text{maximize} \quad Z = 3x_1 + 4x_2 \\ &\text{subject to the conditions} \quad 2x_1 + x_2 \leq 470 \\ &\qquad 3x_1 + 2x_2 \leq 540 \\ &\qquad x_1 \geq 0, \; x_2 \geq 0 \end{aligned}}$$

Example 2 :

A manufacturer of packing material manufactures two types of packing tins round and flat. Major production facilities involved are cutting and Joining. The cutting department can process 300 tins of round type or 500 tins of flat type per hour. The joining department can process 400 tins of round type or 300 tins of flat type per hour. If the profit contribution of round tins is Rs. 100/- per tin and that of flat is Rs. 80/-

per tin. Formulate (only), the problem as linear programming problem. [VTU July 2003, Common Paper]

Solution :

Let x_1 be the number of round tins and x_2 be the number of flat tins.

The information given in the problem can be arranged as shown in the table 2.0.

Table 2.0

	Tins	
	Round (x_1 numbers)	Flat (x_2 numbers)
Processing by cutting dept / hour	300	500
Processing by joining dept / hour	400	300
Profit (Rs) / tin	100	80

Since the processing times available for cutting and joining departments are not given, let us consider a unit time and thus the constraints of cutting and joining departments may be written as follows.

$$\frac{x_1}{300} + \frac{x_2}{500} \leq 1$$, cutting department constraint.

Similarly, $\frac{x_1}{400} + \frac{x_2}{300} \leq 1$, Joining department constraint.

And the non-negativity restrictions are

$x_1 \geq 0$ and $x_2 \geq 0$.

The objective function becomes,

maximize $Z = 100x_1 + 80x_2$.

$\therefore$ The LPP formulation is

Maximize,	$Z = 100x_1 + 80x_2$
Subject to the condition:	$\frac{x_1}{\cancel{3}00} + \frac{x_2}{500} \leq 1$
	$\frac{x_1}{400} + \frac{x_2}{300} \leq 1$
	$x_1 \geq 0, x_2 \geq 0$.

Example 3:

A medicines manufacturer proposes to prepare a production plan for medicines A and B. There are sufficient ingredient available to make 30 thousand bottles of medicine A and 50 thousand bottles of medicine B. But there are only 40 thousand bottles into which either of the medicines can be filled. Further, it takes 4 hours and 3 hours respectively to prepare enough material to fill thousand bottles of medicine A and B respectively and there are only 70 hours available. The profit is Rs. 9 and Rs. 6 per bottle of medicine A and B respectively. Formulate this problem as an LPP.

Solution:

Let the manufacturer produce x_1 and x_2 thousands of bottles of medicines A & B respectively

Since the ingredients available to make medicine A and B respectively are 30 thousand and 50 thousand bottles, the constraints becomes,

$$x_1 \leq 30$$

$$x_2 \leq 50$$

But the number of bottles available are only 40 thousand, thus we can write

$$x_1 + x_2 \leq 40$$

Further, the time restrictions are

$$4x_1 + 3x_2 \leq 70$$

The profits given are Rs. 9 & Rs. 6 per bottle; thus for x_1 and x_2 thousand bottles of medicine A & B, the objective function becomes

$$Z = 9 \times 1000 x_1 + 6 \times 1000 x_2$$

Therefore, the fromulation is

$$\text{maximize } Z = 9000\, x_1 + 6000\, x_2$$

$$\text{STC} \quad 4x_1 + 3x_2 \leq 70$$

$$x_1 + x_2 \leq 40$$

$$x_1 \leq 30$$

$$x_2 \leq 50$$

$$x_1 \geq 0, \; x_2 \geq 0.$$

Example 4:

Two products A & B are made involving two chemical operations for each. The production of B also results in a by product C. The products A & B can be sold at a profit of Rs. 4 and 9 per unit respectively. While the byproduct C can be sold at a profit of ~~but it cannot be sold as the~~ Rs. 2 per unit, ~~destruction cost~~; Forecasts show that upto 5 units of C can be sold. The company gets 3 units of C for each unit of B produced. Each unit of product A requires 2 hours on operation I and 3 hours on operation 2. Each unit of product B requires 3 hours on operation 1 and 4 hours on operation 2. The available times for operation 1 and 2 are 16 hours and 24 hours respectively. The problem is to determine the production quantity of A and B, keeping C in mind, so as to make maximum profit. Formulate as an LP problem.

Solution:

Let x_1, x_2 and x_3 be the number of A, B and C products produced.

The total profit is given by

$$Z = 4x_1 + 9x_2 + 2x_3$$

The company gets 3 units of C for each unit of B produced and upto 5 units of C canbe sold

$$3x_2 \leq 5$$

	Product A	Product B	Available Time
operation 1	2 Hrs	3 Hrs	16 Hrs
operation 2	3 Hrs	4 Hrs	24 Hrs

∴ The time constraints are

$$2x_1 + 3x_2 \leq 16$$
$$3x_1 + 4x_2 \leq 24$$

Thus the formulation is.

Maximize $Z = 4x_1 + 9x_2 + 2x_3$

STC $\cancel{x_2}\ 3x_3 \le 5$

$2x_1 + 3x_2 \le 16$

$3x_1 + 4x_2 \le 24$

$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$

(or) Obj fn $\rightarrow$ max $Z = 4x_1 + 15x_2$.

Example 5 :

A car dealer wishes to stock up his lot to maximize his profit. He can select cars A, B and C which are valued at wholesale price of Rs. 5000, Rs. 7000 and Rs. 8000 respectively. They can be sold at Rs. 6000, Rs. 8500 and Rs. 10500 respectively. For each car type the probabilities of sales are :

	A	B	C
Type of car :			
Probability of sale in 90 days :	0.7	0.8	0.6

For every two cars of B, he should buy one car of type A or C, If he had Rs. 100,000 to invest, what would he buy to maximize his expected gain? Formulate the linear programming problem.

Solution :

Let x_1, x_2 and x_3 be the number of cars of type A, B and C respectively.

The total profit of the dealer is given by

$$Z = (6000-5000)\,0.7x_1 + (8500-7000)\,0.8x_2$$

$$+ (10500-8000)\,0.6\,x_3$$

or $Z = 700x_1 + 1200x_2 + 1500x_3$

And the investment constraints are

$$5000x_1 + 7000\,(2x_2) \leq 100000$$

$$7000(2x_2) + 8000x_3 \leq 100000$$

Thus the formulation is

Max $Z = 700x_1 + 1200x_2 + 1500x_3$

STC $5000x_1 + 14000x_2 \leq 100000$

$14000x_2 + 8000x_3 \leq 100000$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$

Example 6 :

The production manager ~~of a~~ Toy manufacturing company is scheduling hourly production levels for three of its more popular dump trucks, Junior, mighty and Super. Each truck is processed through three departments fabrication, assembly & Packaging.

Table below indicates the time required to process a unit in each department (in minutes per unit), ~~the total production time available during each hour of operation~~ and profit margins for each product.

Department.	Junior	Mighty	Super
Fabrication	1.00	1.20	1.25
Assembly	1.3	1.5	1.6
Packaging	0.8	0.75	0.9
profit per unit	0.55	0.70	0.75

management has specified minimum hourly production levels for the Junior and mighty trucks of 200 and 250 units respectively. Also, no more than 225 units of the Super truck should be produced each hour. The production manager wishes to determine the number of trucks to schedule each hour so as to maximize the total contribution to profit. formulate the manager's problem as a linear programming model.

Solution:

Let x_1, x_2 & x_3 be the number of Junior, mighty and Super trucks respectively, scheduled for production.

Thus the formulation is.

Maximize $Z = 0.55x_1 + 0.7x_2 + 0.75x_3$

STC
$$x_1 + 1.2x_2 + 1.25x_3 \leq 60$$
$$1.3x_1 + 1.5x_2 + 1.6x_3 \leq 60$$
$$0.8x_1 + 0.75x_2 + 0.9x_3 \leq 60$$
$$x_1 \geq 200$$
$$x_2 \geq 250$$
$$x_3 \leq 225$$
$$x_3 \geq 0$$

Example 7:

A boat manufacturer builds two types; Type A and. Type B boats. The boats built during the months 4 January-June go on sale in the months July-December at a profit of Rs. 2000/- per type A boat and Rs. 1500/- per type B boat. Those built during the months July-December go on sale in the months January-June at a profit of Rs. 4000/- per type A boat and Rs. 3300/- per type B boat. Each type A boat requires 5 hours in the carpentry shop and 3 hours in the finishing shop. Each type B boat requires 6 hours in the carpentry shop and 1 hour in the finishing shop. During each half year period a maximum of 12000 hours and 15000 hours are available in the carpentry and finishing shops respectively. Sufficient material is available to build no more than 3000 type A boats and 3000 type B boats a year. How many of each type of boat should be built during each half year in order to maximize the profit. Formulate as an LPP. [Jan 2005]

Solution:

Let x_{A1} and x_{A2} be the number of type 'A' boats built during January - June and July - December respectively.

Similarly, x_{B1} and x_{B2} are the number of type B boats built during January - June and July - December respectively.

The total profit to the boat manufacturer is given by,

$$Z = 2000\, x_{A1} + 4000\, x_{A2} + 1500\, x_{B1} + 3300\, x_{B2}$$

Time restrictions are

$$5x_{A1} + 6x_{B1} \leq 12000$$

$$5x_{A2} + 6x_{B2} \leq 12000$$

$$3x_{A1} + x_{B1} \leq 15000$$

$$3x_{A2} + x_{B2} \leq 15000$$

material restrictions are

$$x_{A1} + x_{A2} \leq 3000$$

$$x_{B1} + x_{B2} \leq 3000$$

And non negativity restrictions.

$$x_{A1}, x_{A2}, x_{B1}, x_{B2} \geq 0$$

Example 8 :

A farmer has 100 acres land. He can sell all tomatoes, lettuce or radishes he can raise. The price he can obtain is Rs 1.00 per kg for tomatoes, Rs 0.75 a head for lettuce and Rs 2.00 per kg for radishes. The average yield per acre is 2000 kgs of tomatoes, 3000 heads of lettuce and 1000 kgs of radish. Fertilizer is available at Rs 0.50 per kg and the amount required per acre is 100 kgs each for tomatoes, and lettuce and 50 kgs for radish. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radish & 6 man days for lettuce. A total of 400 man-days of labour are available at Rs. 20 per man-day. Formulate this problem as a LPP to maximize the farmers total profit.

Solution:

The objective of the problem is to decide how much area should be alloted to each type of crop. Let x_1, x_2 & x_3 acres of land allotted to grow tomatoes, lettuce & radishes respectively. Then the farmer can produce $2000\,x_1$ kgs of tomatoes, $3000\,x_2$ heads of lettuce and $1000\,x_3$ kgs of radish. Thus the total sales of the farmer in Rs. will be

$$= [1.00 \times 2000x_1 + 0.75 \times 3000x_2 + 2.00 \times 1000x_3]$$

$$= [2000x_1 + 2250x_2 + 2000x_3]$$

Fertilizer expenditure in Rs. will be

$$= 0.5[100(x_1 + x_2) + 50x_3]$$

$$= [50x_1 + 50x_2 + 25x_3]$$

Labour expenditure in Rs. will be

$= 20[5x_1 + 6x_2 + 5x_3]$

$= [100x_1 + 120x_2 + 100x_3]$

Thus the farmers net profit in Rs. will be

$Z = [\text{Total sales (in Rs.)}] - [\text{Total expenditure (in Rs.)}]$

$Z = [2000x_1 + 2250x_2 + 2000x_3] -$

$[(50x_1 + 50x_2 + 25x_3) + (100x_1 + 120x_2 + 100x_3)]$

$\therefore Z = 1850x_1 + 2080x_2 + 1875x_3$

But the total area of the land is restricted to 100 acre

$\therefore x_1 + x_2 + x_3 \leq 100$

Also, the restriction on the labour man-days is

$5x_1 + 6x_2 + 5x_3 \leq 400$

Thus the formulation is

max $Z = 1850x_1 + 2080x_2 + 1875x_3$

STC $x_1 + x_2 + x_3 \leq 100$

$5x_1 + 6x_2 + 5x_3 \leq 400$

$x_1, x_2, x_3 \geq 0$

Example 9

A company has two grades of inspectors, I & II, who are assigned for a quality control inspection. It is required that at least 1800 pieces be inspected per 8-hour day. Grade I inspectors can check pieces at the rate of 25 per hour, with an accuracy of 98%. Grade II inspectors checks at the rate of 15 pieces per hour, with an accuracy of 95%. The wage rate of a grade I inspector is Rs. 4 per hour, while that of a Grade II inspector is Rs. 3 per hour. Each time if an error is made by an inspector, the cost to the company is Rs. 2. The company has eight Grade I inspectors and ten Grade-II inspectors for the inspection job. The company wants to determine the optimal assignment of inspectors, which will minimize the total cost of the inspection. Formulate this problem as an LP model.

Solution:

Let x_1 and x_2 are the number of Grade I and Grade II inspectors respectively and the constraints on the number of available inspectors in the company are

$x_1 \leq 8$

$x_2 \leq 10$

But the company requires at least 1800 pieces to be inspected in 8 hour day.

$\therefore 8(25)x_1 + 8(15)x_2 \geq 1800$

or $200x_1 + 120x_2 \geq 1800 \Rightarrow 5x_1 + 3x_2 \geq 45$

To minimize the total cost of inspection, the company would incur two types of costs during inspection. (i) wages paid to the inspector & (ii) the cost of inspection errors.

Therefore the total cost of Grade I inspectors per hour are

$4 + 2(25)(0.02) = \text{Rs. } 5/\text{hour}$

Similarly, for Grade II inspectors

$3 + 2(15)(0.05) = \text{Rs. } 4.5/\text{hour}$

The objective function is to minimize the daily cost of inspection.

$\therefore$ minimize $Z = 8(5x_1 + 4.5x_2)$

or minimize $Z = 40x_1 + 36x_2$

Thus the formulation is

$$\text{minimize } Z = 40x_1 + 36x_2$$

$$\text{STC} \quad x_1 \leq 8$$

$$x_2 \leq 10$$

$$5x_1 + 3x_2 \geq 45$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

Standard form of Linear Programming Problem

The standard form of a linear programming problem with m constraints and n variables can be represented as follows:

Maximize [minimize] $Z = c_1x_1 + c_2x_2 + \cdots c_nx_n$

Subjected to constraints $a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$

$$a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m$$

$$x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0$$

$$b_1 \geq 0, b_2 \geq 0, \cdots, b_m \geq 0$$

The main features of the standard from are

(i) The objective function is of maximization type

(ii) All the constraints are expressed as equations

(iii) All the variables must have non-negativity values

(iv) The right-hand side constants of each constraint equation should be non-negative

Graphical Solution to L.P.P.

Linear programming problems with two variables can be solved by graphical method and the steps involved in graphical solution are as follows:

Step 1: Consider each inequality-constraint as equation.

Step 2: Plot each equation on the graph, as each one will geometrically represent a straight line.

Step 3: Every point on the line will satisfy the equation of the line. If the inequality-constraint corresponding to that line is "$\leq$" then the region below the line lying in the first quadrant (or towards the origin) is shaded. For the inequality constraint with "$\geq$" the region above the line in the first quadrant (or away from the origin) is shaded.

The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region and is shaded.

Step 4 : Choose a convenient value for Z (say $=0$) and plot the objective function line.

Step 5 : Pull the objective function line to the extreme point of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through atleast one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

Step 6 : Read the coordinates of the extreme point(s) selected in step 5, and find the maximum or minimum (as the case may be) value of Z.

Solving LP problems by Graphical method

problem 1 : max $Z = 2x_1 + 3x_2$

STC $2x_1 + x_2 \leq 12$ ... (1)

$x_1 + 3x_2 \leq 15$ ... (2)

$x_1, x_2 \geq 0$

Solution : Constraint equation (1) $2x_1 + x_2 = 12$

Put $x_1 = 0$; $\therefore x_1 = 0$, $x_2 = 12$ thus, the coordinate is $(0, 12)$

Put $x_2 = 0$; $x_1 = 6$, $x_2 = 0$ thus, the coordinate is $(6, 0)$

Represent the above two coordinates on the graph and join them by a straight line as shown in fig.

Constraint equation (2) $x_1 + 3x_2 = 15$,

Put $x_1 = 0$; $x_1 = 0$, $x_2 = 5$ thus, the coordinate is $(0, 5)$

$x_2 = 0$; $x_1 = 15$, $x_2 = 0$ thus, the coordinate is $(15, 0)$

Similarly, represent the coordinates of the second constraint equation on the graph and join them by a straight lines as shown in fig.

To find the corner point Z of optimal solution.

Put $Z = 0$, in the objective function

$\therefore 2x_1 + 3x_2 = 0$

$2x_1 = -3x_2$

$\frac{x_1}{x_2} = -\frac{3}{2}$

Finding the coordinates of Z from eqns (1) & (2)

$2x_1 + x_2 = 12$

Eqn (2) × 2 $\quad 2x_1 + 6x_2 = 30$

$5x_2 = 18$

Subtracting

$\boxed{x_2 = \frac{18}{5}}$

Substituting in equation (1)

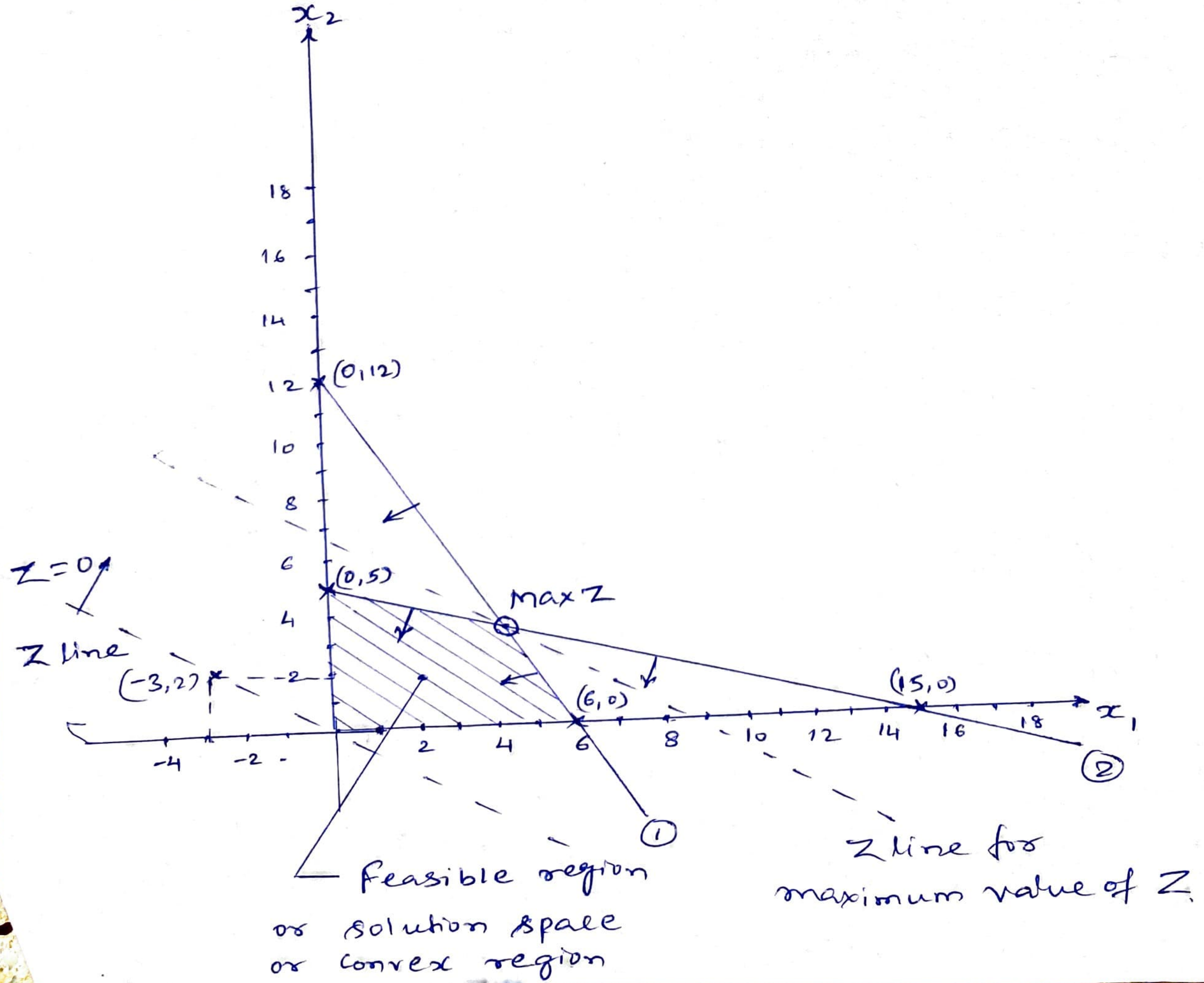
$$2x_1 = 12 - \frac{18}{5} = \frac{60-18}{5} = \frac{42}{5}$$

$$\boxed{\therefore x_1 = \frac{21}{5}}$$

$$\therefore x_1 = \frac{21}{5},\ x_2 = \frac{18}{5}$$

and $Z = 2 \times \frac{21}{5} + 3 \times \frac{18}{5} = \frac{96}{5}$

$$\boxed{\text{Thus, } x_1 = \frac{21}{5},\ x_2 = \frac{18}{5} \text{ and Max. } Z = \frac{96}{5}}$$



Problem 2 : Solve the following L.P.P. by graphica[l]

method :

minimize, $Z = 20x_1 + 10x_2$

Subject to, $x_1 + 2x_2 \leq 40$ ...(1)

$3x_1 + x_2 \geq 30$ ...(2)

$4x_1 + 3x_2 \geq 60$ ...(3)

$x_1, x_2 \geq 0$

Solution :

- Constraint equation (1)

$x_1 + 2x_2 = 40$

Put $x_1 = 0$; $x_2 = 20$ thus $(0, 20)$

$x_2 = 0$; $x_1 = 40$, $x_2 = 0$ thus, $(40, 0)$

~~exa,~~

Constraint equation (2)

$3x_1 + x_2 = 30$

Put $x_1 = 0$; $x_1 = 0$, $x_2 = 30$ thus, $(0, 30)$

$x_2 = 0$; $x_1 = 10$, $x_2 = 0$, thus $(10, 0)$

Constraint equation (3)

$4x_1 + 3x_2 = 60$

Put $x_1 = 0$; $x_1 = 0$, $x_2 = 20$ thus $(0, 20)$

$x_2 = 0$; $x_1 = 15$, $x_2 = 0$ thus $(15, 0)$

Representing all the above coordinates on the graph sheet as shown in the figure.

To find corner point min Z of optimal solution.

Put $Z = 0$, $\therefore 20x_1 + 10x_2 = 0$

$$20x_1 = -10x_2$$

$$\frac{x_1}{x_2} = \frac{-10}{20}$$

Finding the coordinates of minimum Z from equations (2) & (3)

Equation (2) $\times 3 \Rightarrow 9x_1 + 3x_2 = 90$

$$4x_1 + 3x_2 = 60$$

Subtracting $\quad 5x_1 = 30$

$\therefore x_1 = 6$

Substituting $x_1 = 6$ in equation (2)

$$3 \times 6 + x_2 = 30$$

$$x_2 = 30 - 18 = 12$$

Hence $Z = 20(6) + 10(12)$

$$= 120 + 120 = 240$$

Thus $x_1 = 6$, $x_2 = 12$ and $\min Z = 240$.

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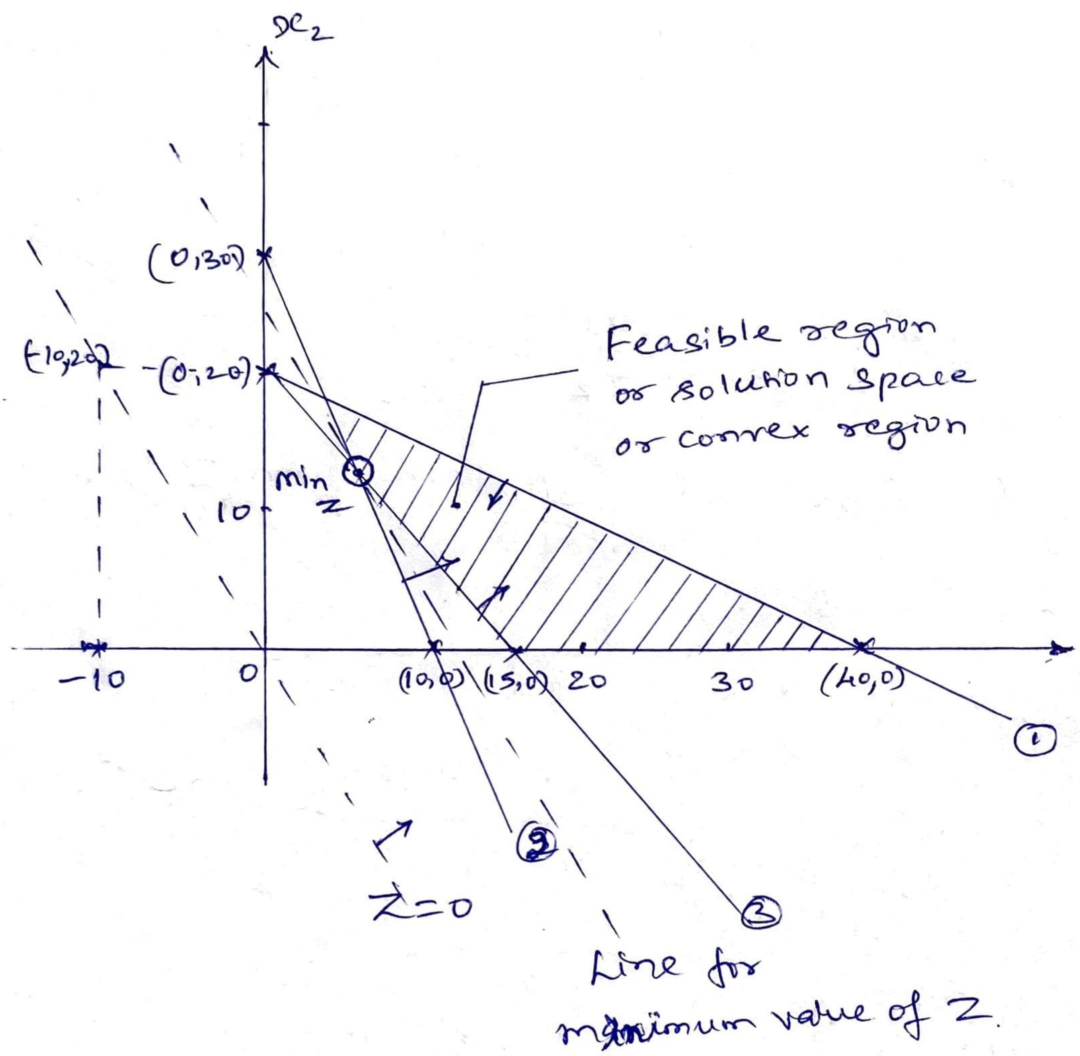


Problem 3 :

Problem with inconsistent system of constraints.

max $Z = 3x_1 + 2x_2$ subject to

$x_1 + x_2 \leq 1$

$2x_1 + 2x_2 \geq 4$

$x_1, x_2 \geq 0$

Solution :

Constraint (1)

$x_1 + x_2 = 1$

Put $x_1 = 0$; $x_1 = 0$, $x_2 = 1$ thus $(0,1)$

$x_2 = 0$; $x_1 = 1$, $x_2 = 0$ thus $(1,0)$

Constraint (2)

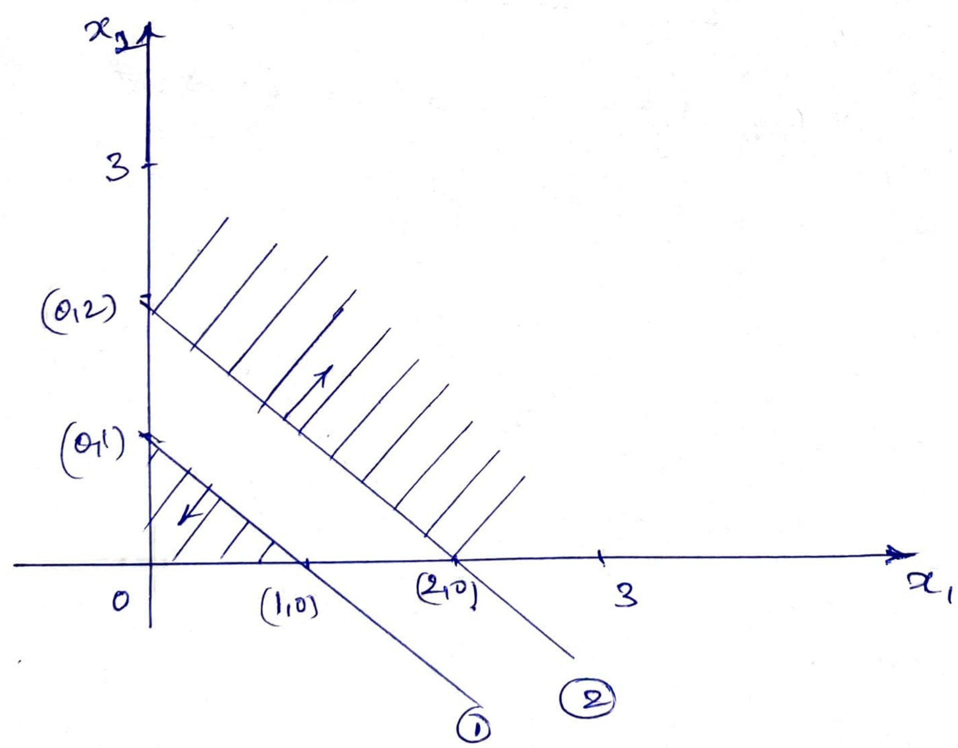
$2x_1 + 2x_2 = 4$

Put $x_1 = 0$; $x_1 = 0$, $x_2 = 2$ thus $(0, 2)$

$x_2 = 0$; $x_1 = 2$, $x_2 = 0$ thus $(2, 0)$

Represent the given constraints as straight lines shown in the fig. 2.5.

From the graph it can be seen that there is no point (x_1, x_2) which satisfies both the constraints simultaneously. Hence the problem has no solution because the constraints are inconsistent.



Problem 4 :

Problem in which constraints are equation rather than inequalities.

Max $Z = 5x_1 + 3x_2$ subject to

$$3x_1 + 5x_2 = 15 \quad \text{-- (1)}$$
$$5x_1 + 2x_2 = 10 \quad \text{-- (2)}$$

Solution :

Eqn (1)

$3x_1 + 5x_2 = 15$

Put $x_1 = 0$; $x_1 = 0$, $x_2 = 3$ thus $(0, 3)$

$x_2 = 0$; $x_1 = 5$, $x_2 = 0$ thus $(5, 0)$

Eqn (2)

$5x_1 + 2x_2 = 10$

Put $x_1 = 0$; $x_1 = 0$, $x_2 = 5$ thus $(0, 5)$

$x_2 = 0$; $x_1 = 2$, $x_2 = 0$ thus $(2, 0)$

Representing the given constraints as straight lines shown in the fig 2.6.

To find point A

Equation (1) $\times 5 \Rightarrow 15x_1 + 25x_2 = 75$

Equation (2) $\times 3 \Rightarrow 15x_1 + 6x_2 = 30$

$19x_2 = 45$

Subtracting

$\therefore x_2 = \frac{45}{19}$

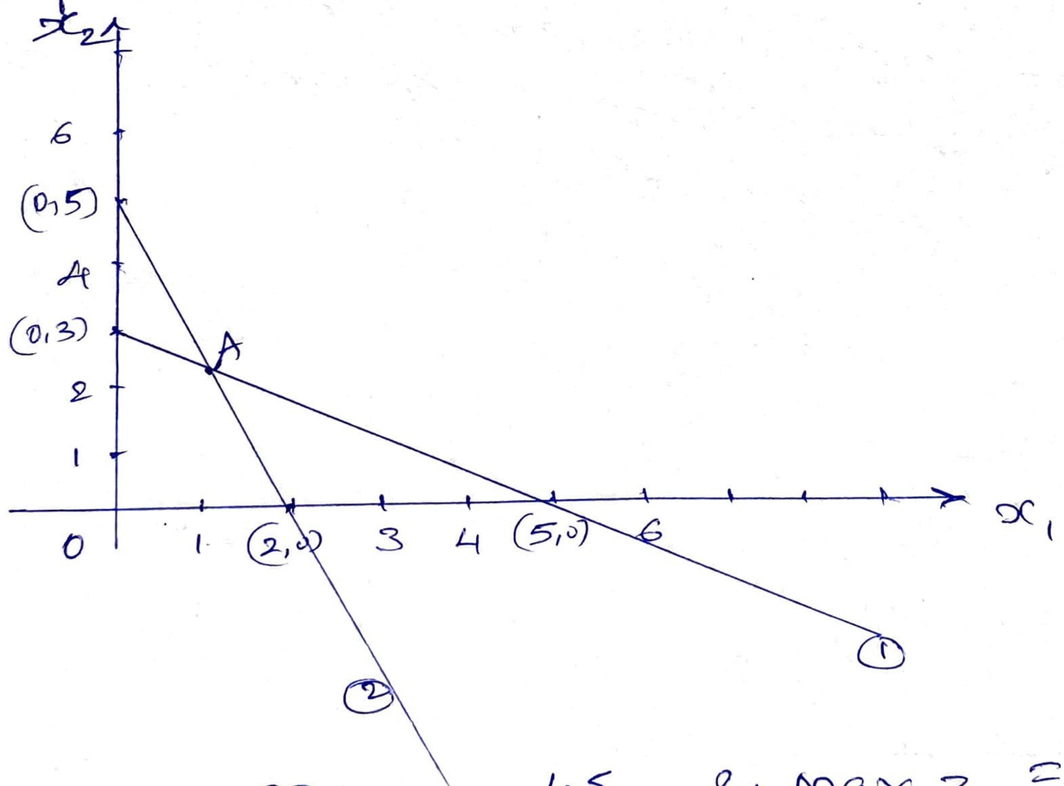
Substitute x_2 in eqn (1)

$$3x_1 + 5 \times \frac{45}{19} = 15$$

$$\therefore x_1 = \frac{20}{19}$$

$$\therefore Z = 5 \times \frac{20}{19} + 3 \times \frac{45}{19} = \frac{100 + 135}{19} = \frac{235}{19}$$

Since there is only a single point A, there is nothing to be maximized. If the solution is feasible, it is optimal. If it is not feasible, the problem has no solution.



Thus $x_1 = \frac{20}{19}$, $x_2 = \frac{45}{19}$ & max $Z = \frac{235}{19}$

Problem 5: Min $Z = x_1 + x_2$

Subject to $5x_1 + 10x_2 \leq 50$ ---(1)

$x_1 + x_2 \geq 1$ ---(2)

$x_2 \leq 4$ ---(3)

$x_1, x_2 \geq 0$

Solution! Constraint (1)

$$5x_1 + 10x_2 = 50$$

put $x_1 = 0$; $x_1 = 0, x_2 = 5$ thus $(0, 5)$

$x_2 = 0$; $x_1 = 10, x_2 = 0$ thus $(10, 0)$

Constraint (2)

$$x_1 + x_2 = 1$$

Put $x_1 = 0$; $x_1 = 0, x_2 = 1$ thus $(0, 1)$

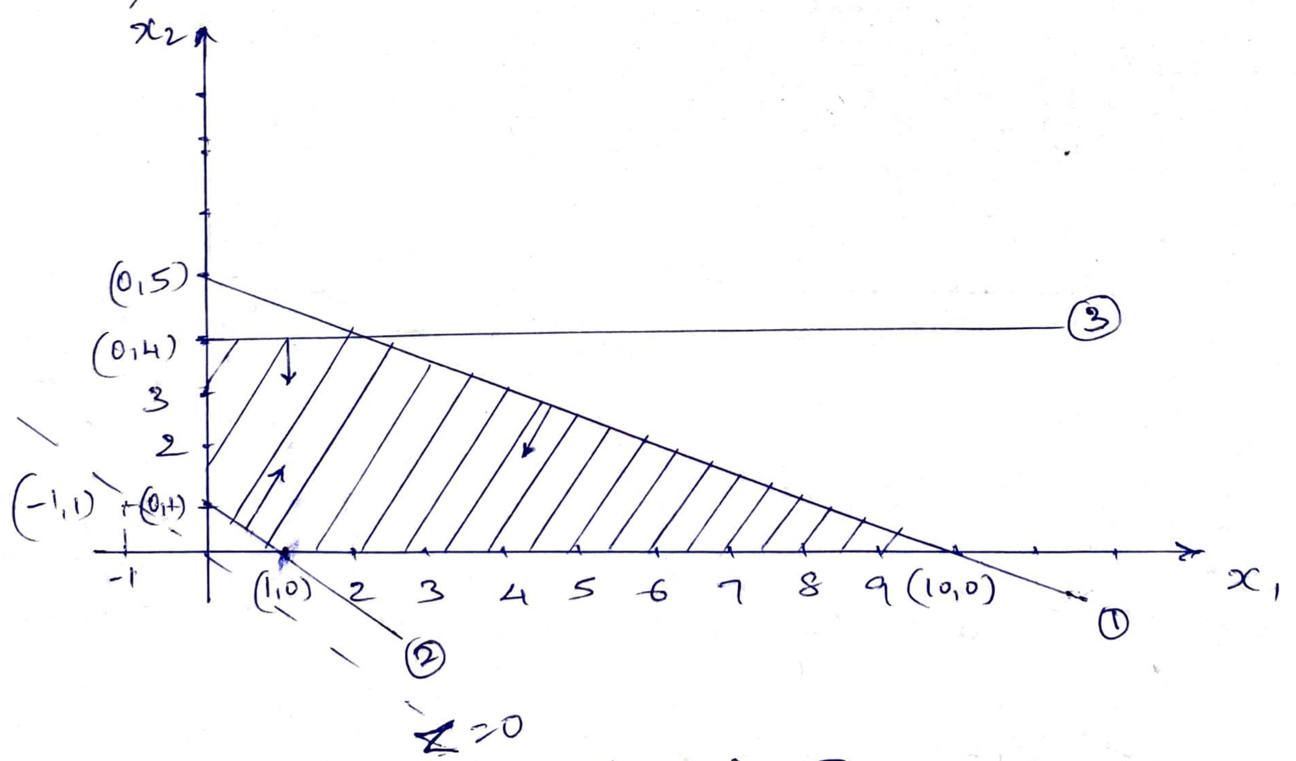
~~$x_2 = 0$; $x_2 = 1$~~

$x_2 = 0$; $x_1 = 1, x_2 = 0$ thus $(1, 0)$

Constraint (3)

$x_2 = 4$ thus $(0, 4)$

Represent all the above constraints as straight lines on the graph as shown in fig ~~2.3~~.



To find corner point min Z.

Put $Z=0$, $x_1 + x_2 = 0$.

$$x_1 = -x_2$$

or $\frac{x_1}{x_2} = -\frac{1}{1}$

The line of objective equation coincides with the boundary line of the polygon of feasible solution. This indicates that the values of x_1 & x_2 which minimize Z are not unique, but any point on this edge of the polygon will give the optimum value of Z. The minimum value of Z is always unique, but there will be an infinite number of feasible solutions which give unique value of Z.

It should be noted that if a linear programming problem has more than one optimum solution, there exists alternative optimum solutions.

To get the value of Z, substitute either (0,1) or (1,0) of the equation (2) in the objective function.

$\therefore Z = 0 + 1 = \underline{1}$

Problem 6 Problem having unbounded solution.

$$\max Z = 3x_1 + 2x_2$$

StC
$$x_1 - x_2 \leq 1$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

Solution:

Constraint (1)

$x_1 - x_2 = 1$

Put $x_1 = 0$; $x_1 = 0$, $x_2 = -1$ thus $(0, -1)$

$x_2 = 0$; $x_1 = 1$, $x_2 = 0$ thus $(1, 0)$

Constraint (2)

$x_1 + x_2 = 3$

Put $x_1 = 0$; $x_1 = 0$, $x_2 = 3$ thus $(0, 3)$

$x_2 = 0$; $x_1 = 3$, $x_2 = 0$ thus $(3, 0)$

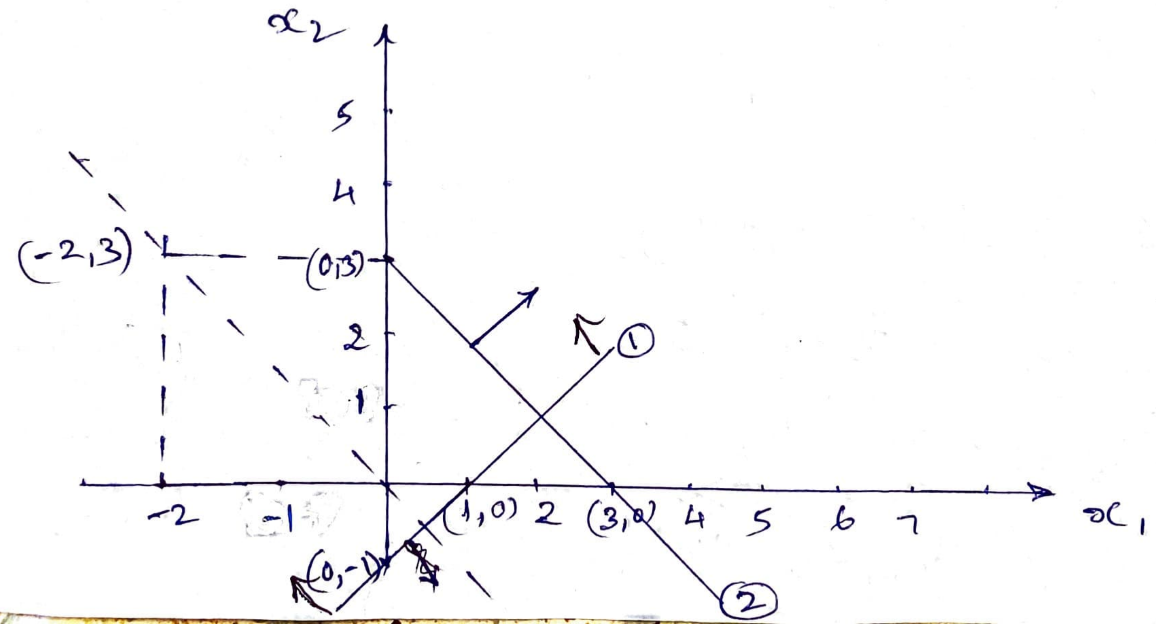
Represent the given constraints as straight lines shown in the fig.

To find Z

Put $Z = 0$, $\therefore 3x_1 + 2x_2 = 0$

$3x_1 = -2x_2$

$\frac{x_1}{x_2} = -\frac{2}{3}$



It is evident from the graph that the line representing the objective function can be moved far even parallel to itself in the direction of increasing Z and still have some points in the region of feasible solutions.

Hence Z can be made arbitrarily large and the problem has no finite maximum value of Z. Such problems are said to have unbounded solution.

Infinite profit in practical problems of LP cannot be expected. If L.P.P has been formulated by committing some mistake, it may lead to an unbounded solution.

Problem 7

max $Z = 3x + 2y$

s.t.c $-2x + 3y \leq 9$ --- (1)

$x - 5y \geq -20$ --- (2)

$x, y \geq 0.$

Solution:

Constraint (1)

$-2x + 3y = 9.$

Put $x = 0$; $x = 0, y = 3$ thus $(0, 3)$

$y = 0$; $x = -\frac{9}{2}$, $y = 0$ thus $(-\frac{9}{2}, 0)$

Constraint (2).

$x - 5y = -20$

Put $x = 0$; $x = 0, y = 4$ thus $(0, 4)$

$y = 0$; $x = -20, y = 0$ thus $(-20, 0)$

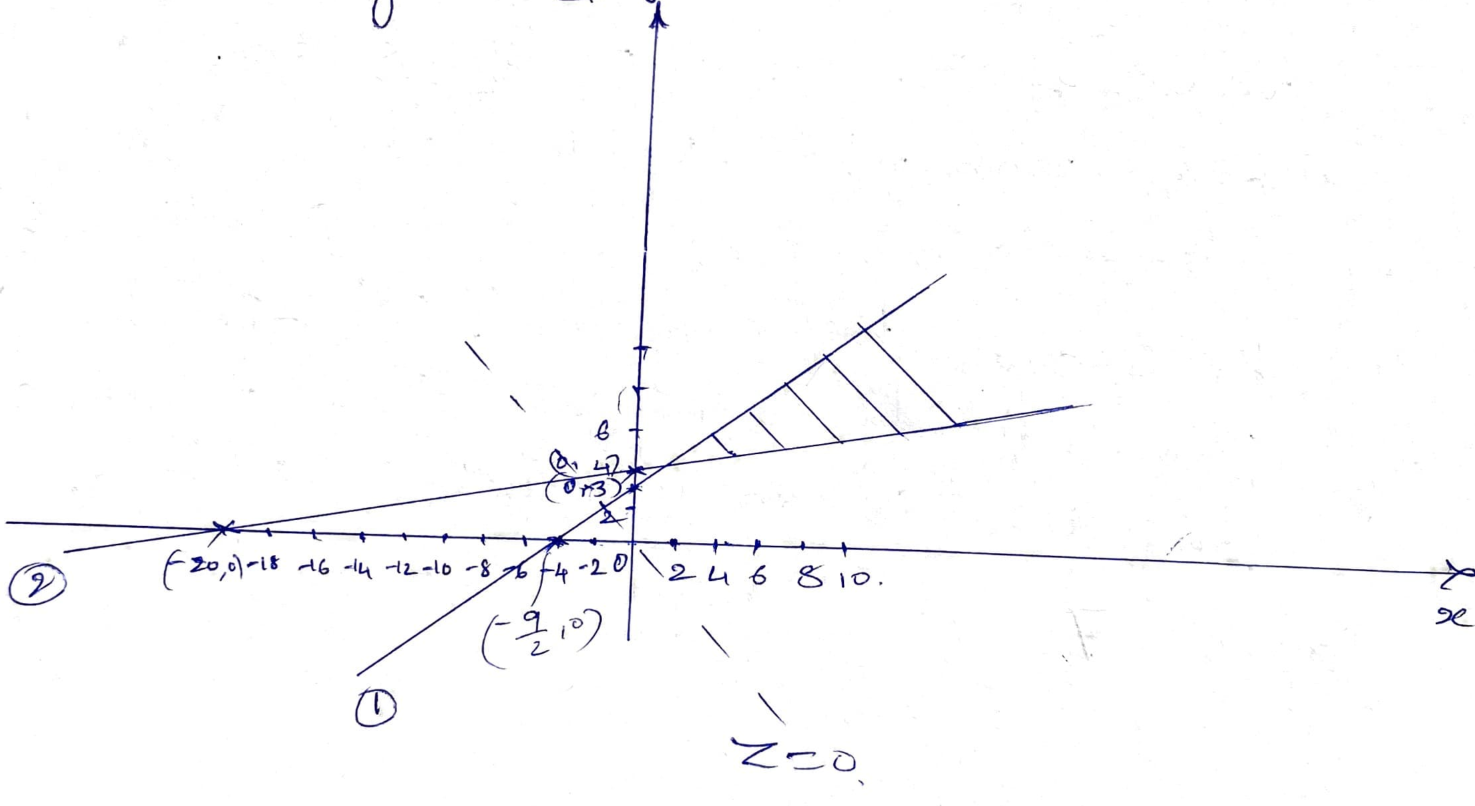
Represent the given constraints as straight lines shown in the figure.

To find Z.

Put $Z=0$, $3x+2y=0$.

$3x = -2y$

$\frac{x}{y} = -\frac{2}{3}y$



Solution is unbounded.

Problem. 8

old machines can be bought at Rs 2 lakhs each & new machines at Rs 5. lakhs each. The old machines produce 3 components / week while the new machines produce 5 components / week, each component being worth Rs. 30,000. A machine (new or old) costs Rs 1 lakh / week to maintain. The company has only Rs. 80 lakhs to spend on machines. How many of each kind should the company buy to get a profit of more than Rs. 6 lakhs/week. Assume that the company cannot house more than 20 machines.

Solution: The given data can be written as follows.

	old	New	
Buying Price (lakhs)	2	5	≤ 80
Production/week	3	5	worth 30,000 (0.3 lakhs)
maintenance Cost/week	1	1	≤ 20
	x_1	x_2	

Let x_1 and x_2 be the number of old and new machines respectively.

Since old machines produce 3 Components/wk and new m/cs can produce 5 Components/wk, the total number of Components/wk $= 3x_1 + 5x_2$

Consequently, the cost of each component being Rs. 30,000, the total gain will be

$= 0.3(3x_1 + 5x_2)$ in lakhs

Total expenditure for maintaining $(x_1 + x_2)$ machines at the rate of Rs 1 lakh each will be

$= 1(x_1 + x_2)$ in lakhs

Thus the total profit Z earned / week will be Z = total gain - total expenditure

$Z = 0.3(3x_1 + 5x_2) - (x_1 + x_2)$

$Z = -0.1x_1 + 0.5x_2$

Since old machines can be brought at Rs. 2 lakhs each and new machines at 5 lakhs each and there are only Rs. 80 lakhs available for purchasing,

$2x_1 + 5x_2 \leq 80.$

Also, since it is not possible to house more than 20 machines,

$x_1 + x_2 \leq 20.$

Also since the profit is restricted to be more than Rs. 6 lakhs, this means that the profit function Z is to be maximized. Thus, there is no need to add one more constraint ie., $-0.1x_1 + 0.5x_2 \geq 6.$

And $x_1 \geq 0,\ x_2 \geq 0.$

Therefore max $Z = -0.1x_1 + 0.5x_2$

s.t.c. $2x_1 + 5x_2 \leq 80$ --- (1)

$x_1 + x_2 \leq 20$ --- (2)

$x_1, x_2 \geq 0,$

Equation (1) => $2x_1 + 5x_2 = 80$

$x_1 = 0$; $x_1 = 0, x_2 = 16$ thus $(0, 16)$

$x_2 = 0$; $x_1 = 40, x_2 = 0$ thus $(40, 0)$

Equation (2) => $x_1 + x_2 = 20$.

$x_1 = 0$; $x_1 = 0$; $x_2 = 20$ thus $(0, 20)$

$x_2 = 0$; $x_1 = 20, x_2 = 0$ thus $(20, 0)$

Representing the above coordinates as shown in the figure.

To find Z.

Put $Z = 0$, $\therefore -0.1x_1 + 0.5x_2 = 0$.

$0.5x_2 = 0.1x_1$

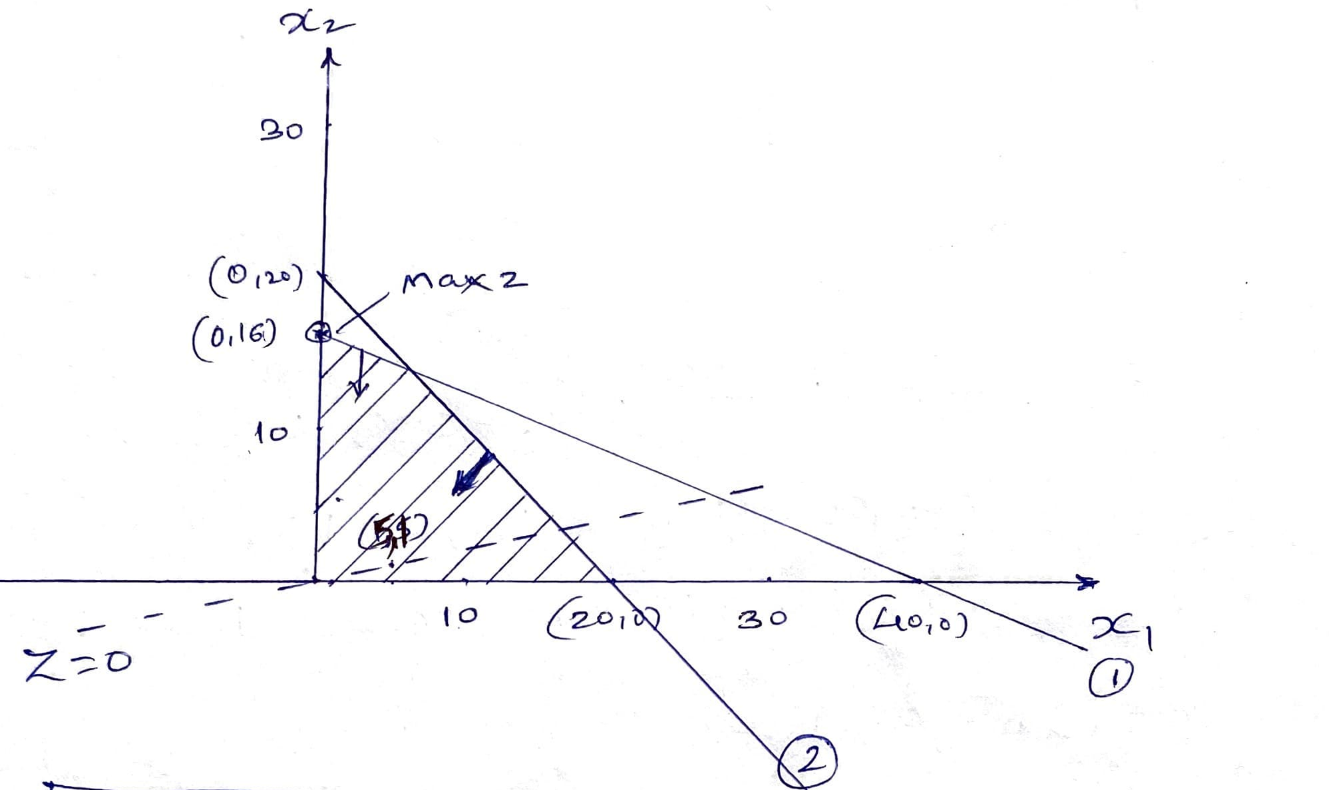
$$\frac{x_2}{x_1} = \frac{0.1}{0.5} = \frac{1}{5}$$

From the figure the maximum value of Z is at the coordinate $(0, 16)$

Thus, $x_1 = 0$, $x_2 = 16$

$\therefore Z = -0.1(0) + 0.5(16)$

$\therefore \underline{\underline{Z = 8}}$



$$\boxed{x_1 = 0, \quad x_2 = 16 \quad \& \quad \max Z = 8}$$

Hence only 16 new machines should be brought in order to get the maximum profit of Rs. 8 lakhs

Exam Question : June / July 2015 10 marks

1 (b). The XYZ Company has been a producer of electronic circuits for Television sets and certain printed circuit boards for Radios. The company has decided to expand into full scale production and marketing of Am and Am-Fm radios. It has built a new plant that can operate 48 hours per week. production of an Am radio in the new plant will require 2 hours and production of Am. Fm radio will require 3 hours. Each Am radio will contribute Rs. 40 to profit, while

an Am-Fm radio will contribute Rs. 80 to profits. The marketing department, after extensive research, has determined that a maximum of 15 Am radio, and 10 Am-Fm radios can be sold each week. Formulate a L.P. model to determine the optimal production mix of Am and Am-Fm radios that will maximize profits and solve the problem using Graphical method.

New plant can operate - 48 hrs /wk.

	Prdn hrs	Profit	max sales
Am Radio	- 2 hrs	Rs. 40	15
Am. Fm Radio	- 3 hrs.	Rs. 80	10

Let x_1 - number of Am radios to manufacture.
x_2 - number of Am.Fm radios to manufacture.

obj. fn Max. Z = $40x_1 + 80x_2$
Profit.

Stc., $2x_1 + 3x_2 \leq 48$

$0 \leq x_1 \leq 15$

$0 \leq x_2 \leq 10$

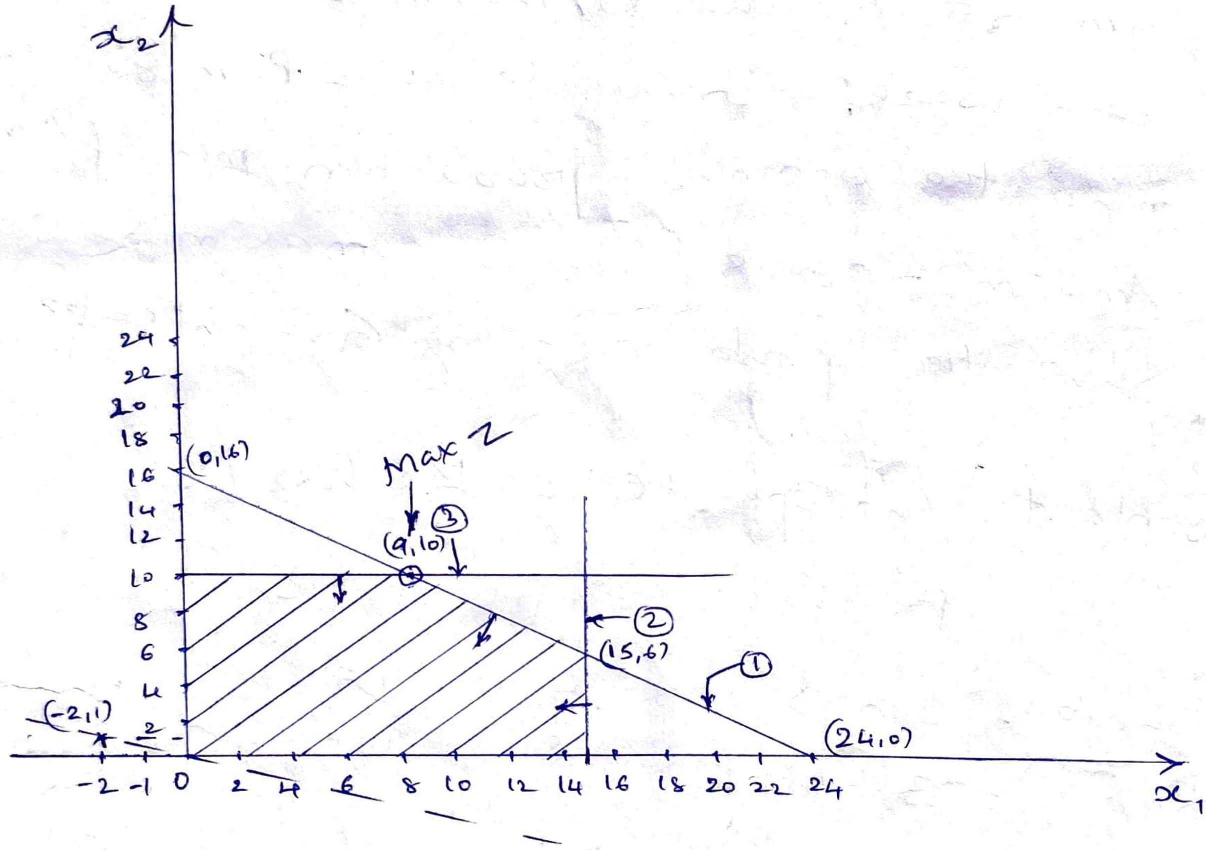
Solving using Graphical method.

$2x_1 + 3x_2 = 48$

$x_1 = 0, \quad x_2 = 16$
$x_2 = 0, \quad x_1 = 24$

obj fn - $40x_1 = -80x_2$

$$\frac{x_1}{x_2} = -\frac{80}{40} = -\frac{2}{1}$$



$x_2 = 10$

$2x_1 + 3x_2 = 48$

$\cancel{2(x10)}$

$2x_1 + 3(10) = 48$

$2x_1 = 18$

$x_1 = 9$

$$\max Z = 40x_1 + 80x_2$$

$$= 40(9) + 80(10)$$

$$= 1{;}160$$