Discrete Mathematical Structures
(CS330)

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Course Outcomes

CO1 : Knowledge of the concepts needed to test the logic of the program.

CO2 : Demonstrate the understanding of relations and able to determine the properties.

CO3 : Perceive, construct and decode group codes based on the various methods.

CO4 : Demonstrate different traversal methods for trees and graphs.

Course Contents

Unit No.	Course Content	No. of Hours
1.	**Fundamental of Logic:** Basic Connectives and Truth Tables, Logical Equivalence: The Laws of Logic, Logical Implication: Rules of Inference, The Use of Quantifiers, Quantifiers, Definitions and the Proofs of Theorems.	08
2.	**Relations:** Properties of relations, Computer Recognition: Zeros- One Matrices and Directed Graphs, Partial; orders: Hasse diagrams , Equivalence Relations and Partitions , Lattices.	08
3	Elements of coding theory and Hamming Metric, generation of codes using Parity check and Generator matrices.	08
4.	**Graph theory:** Definitions and Examples, Subgraphs, Complements, and Graph Isomorphism, Vertex degree, Euler trail and circuits, Planar Graphs, Hamiltonian Paths and cycles.	08
5.	**Trees:** Definitions, Properties and Examples, Rooted trees, trees and sorting weighted trees and prefix Codes.	07

Text and Reference Books

Text Book:

1. Ralph.P.Grimaldi, B.V.Ramana, “Discrete and Combinatorial Mathematics”, 5th Edition, Pearson Education -2009.

Reference Books:

1. Kenneth. H.Rosen, “Discrete Mathematical Structures Theory and Application”, V Edition, PHI/Pearson, Education, 2004.
2. Kolman, Busby and Ross, “Discrete Mathematical Structures”, Fourth Edition, Prentice –Hall of India Pvt Ltd-2009.

What is Discrete Mathematics?

Discrete mathematics is a study of discrete structures which are abstract mathematical models that deals with discrete objects and their relationships.

Applications of Discrete Mathematics

- Checking the correctness of a program
- Computer Networks
- Operating System
- Finite Automata
- Artificial Intelligence
- Algorithm Design etc.,

Unit – 1

Fundamentals of Logic

What is Logic ?

- Logic deals with the methods of reasoning
- Logic is extremely useful in decision making problems
- Logic theory forms the basics for Artificial intelligence, Fuzzy logic, Expert Systems etc.,

Propositions

- A *proposition* is a statement or sentence that can be determined to be either true or false (but no both).
- Examples:
  - The only positive integers that divide 7 are 1 and 7 itself. (True)
  - Everybody likes chocolate.(False)
  - It is raining outside (True or False)

Continued…

Not all sentences are proposition

Eg.,

- $x + 3 > 7$ /* we cannot decide whether True or False unless we know the value of x */
- Give your book /* It's a sentence but not statement , hence its not a proposition */
- What a beautiful garden ! /* exclamation and commands are sentences but not statements */

Example

- Use variable(p, q, r..) to represent propositions

- p: 1+1=3
- q: It is raining outside
- r: Today is Tuesday

The statements represented by p, q, r are considered to be primitive statements because they cannot be broken down into anything simpler

Connectives

If p and q are propositions, new *compound* propositions can be formed by using *connectives*

- Most common connectives:
  - Conjunction (and) $\wedge$
  - Disjunction (or) $\vee$
  - Negation (not) $\sim$ ($\neg$)
  - Exclusive-OR $\underline{\vee}$
  - Condition (if … then) $\rightarrow$
  - Bi-Condition $\leftrightarrow$

Example

- P: It is raining
- Q: It is cold

- Form a new compound statement by combining these two statements

  $P \wedge Q$: It is raining *and* it is cold

  $P \vee Q$: It is raining *or* it is cold

Truth table of conjunction

- The truth values of compound propositions can be described by *truth tables.*
- Truth table of *conjunction*

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

- $P \wedge Q$ is true only when both P and Q are true.

Example

- Let P = "A decade is 10 years"
- Let Q = "A millennium is 100 years"
- $P \wedge Q$ = "A decade is 10 years" and "A millennium is 100 years"
- If P is true and Q is false then conjunction is false

Truth table of disjunction

- The truth table of *disjunction* is

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

- p ∨ q is false only when both p and q are false
  - Example: p = "John is a programmer", q = "Mary is a lawyer"
  - p v q = "John is a programmer or Mary is a lawyer"

Truth table of Negation

- Negation of P: in symbols ~P or ⌐P

P	~P
T	F
F	T

- ~P is false when P is true, ~P is true when P is false

Negation Continued…

- E.g
- Example, P : "John is a programmer"

  ~P = "John is not a programmer"

- P: Paris is the capital of England
- ~P: Paris is not capital of England

Exclusive disjunction

- "Either P or Q" (but not both), in symbols $P \veebar Q$

P	Q	$P \veebar Q$
T	T	F
T	F	T
F	T	T
F	F	F

- $P \veebar Q$ is true only when P is true and Q is false, or P is false and Q is true.
  - Example: p = "John is programmer, q = "Mary is a lawyer"
  - $p \veebar q$ = "Either John is a programmer or Mary is a lawyer"

More compound statements

- Let p, q, r be simple statements
- We can form other compound statements, such as

$(p \vee q) \wedge r$

$p \vee (q \wedge r)$

$(\sim p) \vee (\sim q)$

$(p \vee q) \wedge (\sim r)$

and many others…

Example: truth table of $(P \vee Q) \wedge R$

P	Q	R	$(P \vee Q) \wedge R$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

Conditional (Implication)

- A *conditional* proposition is of the form
  "If P then Q"
- In symbols: $P \rightarrow Q$
- Example:

  P = " A bottle contains acid"

  Q = "A bottle has a label"

  $P \rightarrow Q$ = "If a bottle contains acid then it has a label "

Truth table of $P \to Q$

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

$P \to Q$ is true when both p and q are true or when P is false

Example

- If the mathematics department gets an additional amount of Rs. 40,000 then it will hire one new faculty member.

- Let

  P: The Mathematics Department gets an additional Rs. 40,000

  Q: The mathematics Department will hire one new faculty member.

Hypothesis and conclusion

- In a conditional proposition $P \rightarrow Q$,

  P is called the *hypothesis*

  Q is called the *conclusion*

Example

- For all real number x if $x > 0$ then $x^2 > 0$
- P: $x>0$
- Q: $x^2 > 0$

For example $x=3$, $3 > 0$ then $3^2 > 0$ both are true.

$P \rightarrow Q = T \rightarrow T = T$

$x = -2$, $-2 > 0$ is false but $-2^2 > 0$ is true

$P \rightarrow Q = F \rightarrow T = T$

Bi-Conditional

- The *double implication* “p if and only if q” is defined in symbols as $p \leftrightarrow q$

p	q	$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

$p \leftrightarrow q$ is logically equivalent to $(p \rightarrow q)\wedge(q \rightarrow p)$

Exercise Problems

Design a truth table for given compound statement

$$(p \rightarrow q) \wedge [(q \wedge \neg r) \rightarrow (p \vee r)]$$

$(p \rightarrow q) \land [(q \land \neg r) \rightarrow (p \lor r)]$

p	q	r	$p \rightarrow q$ (a)	$q \land \neg r$	$p \lor r$	$[(q \land \neg r) \rightarrow (p \lor r)]$ (b)	$a \land b$
0	0	0	1	0	0	1	1
0	0	1	1	0	1	1	1
0	1	0	1	1	0	0	0
0	1	1	1	0	1	1	1
1	0	0	0	0	1	1	0
1	0	1	0	0	1	1	0
1	1	0	1	1	1	1	1
1	1	1	1	0	1	1	1

Exercise Problems continues…

Let s, t and u denote the following primitive statements

s : Phyllis goes out for walk

t : The moon is out

u : It is snowing

Express the following compound proposition in words

a) $(t \wedge \neg u) \rightarrow s$

b) $t \rightarrow (\neg u \rightarrow s)$

c) $\neg(s \leftrightarrow (u \vee t))$

Solution

a) If the moon is out and it is not snowing, then Phyllis goes out for walk.
b) If the moon is out then if it is not snowing then Phyllis goes out for a walk.
c) It is not that Phyllis goes out for a walk if and only if it is snowing or the moon is out.

Exercise problems continues…

Write the compound proposition for the following sentences.

d) Phyllis will go out for walking if and only if the moon is out. ($s \leftrightarrow t$)

e) If it is snowing and moon is not out, then Phyllis will not go out for walk ($(u \wedge \neg t) \rightarrow \neg s$)

f)It is snowing but Phyllis will still go out for a walk. ($u \wedge s$)

Exercise problems continues…

Let p and q be primitive statement for which the conditional $p \rightarrow q$ is false.

Determine the truth values of the following compound propositions.

1. $p \land q$
2. $(\neg p \lor q)$
3. $q \rightarrow p$
4. $\neg q \rightarrow \neg p$

Exercise problems continues…

Find the possible truth values of p, q and r in the following cases

1. $p \rightarrow (q \vee r)$ is false
2. $p \wedge (q \rightarrow r)$ is true

Exercise problems continues…

If a proposition q has the true value 1, determine all the truth assignment for the primitive proposition p, q, r and s for which the truth value of the following proposition is 1

$$[q \rightarrow \{(\neg p \vee r) \wedge \neg s\}] \wedge \{\neg s \rightarrow (\neg r \wedge q)\}$$

Solution :

17

$$\underbrace{[q \rightarrow \{(\neg p \vee r) \wedge \neg s\}]}_{x} \wedge \underbrace{\{\neg s \rightarrow (\neg r \wedge q)\}}_{y}$$

$$x \wedge y = 1 \text{ (true)}$$

$x \wedge y = 1$ only when $x = 1$ and $y = 1$

Consider $x = [q \rightarrow \underbrace{\{(\neg p \vee r) \wedge \neg s\}}_{x_1}]$

p	q	p → q
0	0	1
0	1	1
1	0	0
1	1	1

if $q = 1$ (given)

$$\therefore q \rightarrow \underbrace{\{(\neg p \vee r) \wedge \neg s\}}_{x_1 = 1}$$

for x_1 to be 1 $(\neg p \vee r) = 1$ and $\neg s = 1$ as its conjunction.

$\therefore$ If $\neg s = 1$, then $\boxed{s = 0}$

Now, Consider $y = \neg s \rightarrow (\neg r \wedge q)$

$1 \rightarrow \underbrace{(\neg r \wedge q)}$ should be equal to 1.

We know $q = 1$, therefore for $\neg r \wedge q$ to be equal to 1

$\neg r = 1$. $\therefore \boxed{r = 0}$

$$\therefore y = 1 \rightarrow (\neg 0 \wedge 1)$$
$$1 \rightarrow (1 \wedge 1)$$
$$1 \rightarrow 1 = 1$$

$$\therefore \boxed{y = 1}$$

Solution…

18

known $r = 0$, $s = 0$

Again consider..

$$x = [q \rightarrow \{(\neg p \vee r) \wedge \neg s\}]$$

should to 1.

as $r = 0$, for $(\neg p \vee r)$ to be 1

$\neg p$ should be 1, $\therefore$ $p = 0$

$$x = [q \rightarrow \{(\neg 0 \vee 0) \wedge \neg 0)\}$$

$$q \rightarrow \{(1 \vee 0) \wedge 1)\}$$

$$q \rightarrow \{(1 \wedge 1)\}$$

$$q \rightarrow 1$$

$$x = 1 \rightarrow 1 = 1$$

$$x \wedge y = 1 \wedge 1 = 1$$

therefore $p = 0$, $r = 0$ & $s = 0$.

Tautology

- A proposition is a *tautology* if its truth table contains only true values for every case
  - Example: $p \rightarrow p \vee q$

p	**q**	**$p \rightarrow p \vee q$**
T	T	T
T	F	T
F	T	T
F	F	T

Contradiction

- A proposition is a *Contradiction or absurdity* if its truth table contains only false values for every case
  - Example: p ^ ~p

p	p ^ (~p)
T	F
F	F

Contingency

A compound proposition that can be true or false (depending upon the truth values of its components) is called Contingency.

i.e contingency is a compound proposition that is neither tautology or contradiction

Exercise problems :

1. Prove that, for any proposition p,q,r the following compound propositions are Tautology

a) $[(p \rightarrow q) \Lambda (q \rightarrow r)] \rightarrow (p \rightarrow r)$

b) $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$

2. Show that $[(p \wedge q) \wedge r] \rightarrow (s \vee t)$ is contradiction

Logical equivalence ($\Leftrightarrow$)

- Two propositions are said to be *logically equivalent* $\Leftrightarrow$ if their truth tables are identical.

P	Q	$P \rightarrow Q$	$\sim P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

- Example: $P \rightarrow Q$ is *logically equivalent* to $\sim P \vee Q$

$$P \rightarrow Q \Leftrightarrow \sim P \vee Q$$

Problems:

1. Prove that $[(p \vee q) \rightarrow r] \Leftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$
2. $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ are not logically equivalent

Definitions

- Implication $p \rightarrow q$
- contrapositive, $\neg q \rightarrow \neg p$
- converse, $q \rightarrow p$
- inverse $\neg p \rightarrow \neg q$

Table 2.13

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$q \rightarrow p$	$\neg p \rightarrow \neg q$
0	0	1	1	1	1
0	1	1	1	0	0
1	0	0	0	1	1
1	1	1	1	1	1

Converse

- The *converse* of p → q is q → p

p	q	p → q	q → p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

These two propositions
are not logically equivalent

Contrapositive

- The *contrapositive* of the proposition **p → q is ~q → ~p.**

p	q	p → q	~q → ~p
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

They are logically equivalent.

The Laws of Logic

We can prove whether the given Compound proposition is Tautology, Contradiction , contingency or logically equivalent using Truth table method.

But this method becomes tedious when the number of primitive proposition involved is more.

Hence easiest method is to prove using the **Laws of Logic**

1. De Morgan's laws for logic

- The following pairs of propositions are logically equivalent:

  $\sim (p \vee q)$ and $(\sim p) \wedge (\sim q)$

  $\sim (p \wedge q)$ and $(\sim p) \vee (\sim q)$

Proof :

$\sim (p \vee q)$ and $(\sim p) \wedge (\sim q)$

P	q	P ∨ q	~(p ∨q)	~p	~q	~(p) ∧~(q)
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Therefore $\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$

2. Distributive Law

- $p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r)$
- $p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r)$

Laws of Logic

- Let P , Q and R be the propositional variables. T and F represent the True and False respectively.

❖ $\neg\neg P \Leftrightarrow P$	**Law of double negation**
❖ $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ ❖ $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	**De Morgan's laws**
❖ $(P \vee Q) \Leftrightarrow (Q \vee P)$ ❖ $(P \wedge Q) \Leftrightarrow (Q \wedge P)$	**Commutative Laws**
❖ $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$ ❖ $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$	**Associative Laws**

Laws of Logic….

- $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ — **Distributive Laws**
- $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

- $(P \wedge P) \Leftrightarrow P$ — **Idempotent Laws**

  $(P \vee P) \Leftrightarrow P$

- $(P \vee F) \Leftrightarrow P$ — **Identity Laws**
- $(P \wedge T) \Leftrightarrow P$

- $(P \vee \neg P) \Leftrightarrow T$ — **Inverse Laws**
- $(P \wedge \neg P) \Leftrightarrow F$

Laws of Logic…

- $(P \lor T) \Leftrightarrow T$ — **Domination Laws**
- $(P \land F) \Leftrightarrow F$

- $P \lor (P \land Q) \Leftrightarrow P$ — **Absorption Laws**
- $P \land (P \lor Q) \Leftrightarrow P$

- $P \rightarrow Q \Leftrightarrow \neg P \lor Q$ — **Implication Law**

Exercise Problems

Simplify the following compound proposition using the laws of logic:

1. (p v q) ʌ ¬ {(¬p) v q}

23

i) $(p \vee q) \wedge \neg\{(\neg p) \vee q\}$

$= (p \vee q) \wedge \{\neg\neg(p) \wedge \neg q\}$ De Morgan's law

$= (p \vee q) \wedge (p \wedge \neg q)$ Law of double negation

$= [(p \vee q) \wedge p] \wedge (\neg q)$ Association Law.

$= (p \wedge (p \vee q)) \wedge \neg q$ Commutative law.

$= p \wedge \neg q$ Absorption Law.

Exercise Problems

2. Prove that $(p \vee q) \wedge \neg(\neg p \wedge q) \Leftrightarrow p$

2) prove that

37

$(p \vee q) \wedge \neg(\neg p \wedge q) \Longleftrightarrow p$

$= (p \vee q) \wedge [\neg(\neg p) \vee \neg q]$ Demorgan's law

$= (p \vee q) \wedge (p \vee \neg q)$ Law of double negation

$= p \vee (q \wedge \neg q)$ – Distribution law

$= p \vee (F_0)$ – Inverse law

$= p \vee F_0$ – Identity law

$= p$

Exercise Problems

3. Prove that $\neg[\neg\{(p \vee q) \wedge r\} \vee \neg q] \Leftrightarrow q \wedge r$

$= \underline{\underline{P}}$

3) P.T.

$\neg[\neg\{(p \vee q) \wedge r\} \vee \neg q] \Longleftrightarrow q \wedge r.$

$= \neg\neg\{(p \vee q) \wedge r\} \wedge \neg\neg q$

$= (p \vee q) \wedge r \wedge q$ Associative la

$= (p \vee q) \wedge q \wedge r.$ Absorption law

$= \underline{\underline{q \wedge r}}$

Exercise Problems

38

4. $[p \vee q \vee (\neg p \wedge \neg q \wedge r)] \Leftrightarrow (p \vee q \vee r)$

5. $(p \vee q) \rightarrow r \Leftrightarrow (p \rightarrow r) \wedge (q \rightarrow r)$

6. $(p \vee q) \rightarrow (q \rightarrow q)$ is tautology

7. $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow (p \wedge q)$

Exercise Problems

Prove that :-

$$[(P \lor Q) \land \neg\{\neg P \land (\neg Q \lor \neg R)\}] \lor (\neg P \land \neg Q) \lor (\neg P \land \neg R)$$

is tautology

Exercise Problem :

- Give reason for each step in the following simplifications of the compound statement.

(b) $(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)]$

$\Leftrightarrow (p \rightarrow q) \wedge \neg q$

$\Leftrightarrow (\neg p \vee q) \wedge \neg q$

$\Leftrightarrow \neg q \wedge (\neg p \vee q)$

$\Leftrightarrow (\neg q \wedge \neg p) \vee (\neg q \wedge q)$

$\Leftrightarrow (\neg q \wedge \neg p) \vee F_0$

$\Leftrightarrow \neg q \wedge \neg p$

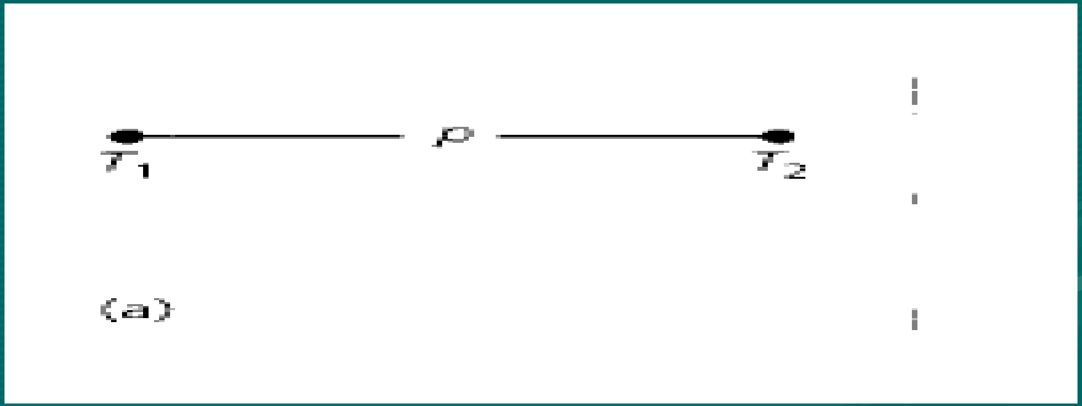
$\Leftrightarrow \neg (q \vee p)$

Application to Switching Networks

- A switching network is made up of wires and switches connecting two terminals say T1 and T2.
- In such a network , each switch is open (so no current flows through it) or closed(so that current flows through it).
- If a switch is open, we assign the symbol 0 to it and if it closed we assign 1.

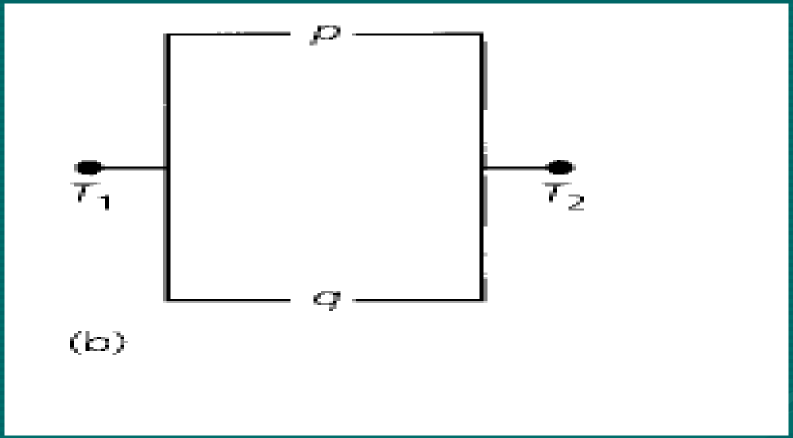
Switching Networks…

- There is a close analog between switches and their states and proposition and their truth values.



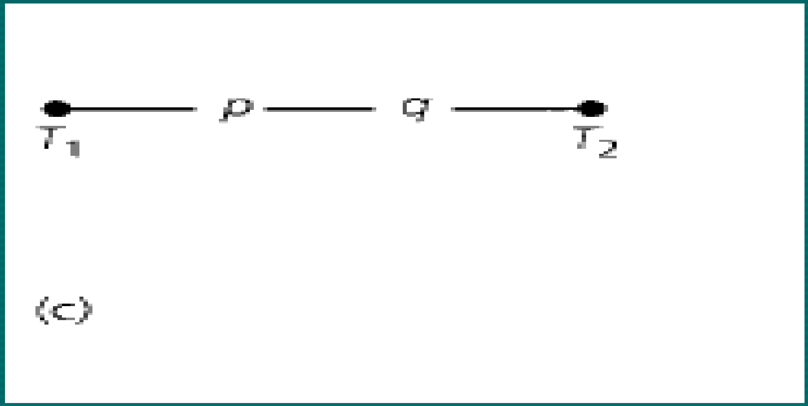
- This shows a network with only one switch p, current flows from terminal T1 to T2 only if switch p is closed. i,e if p is 1

Switching Networks…



- This is a parallel network consisting of two switches p and q in which the current flows from T1 to T2 if either p or q or both are closed.
- Therefore this network can be represented by p v q

Switching Networks…



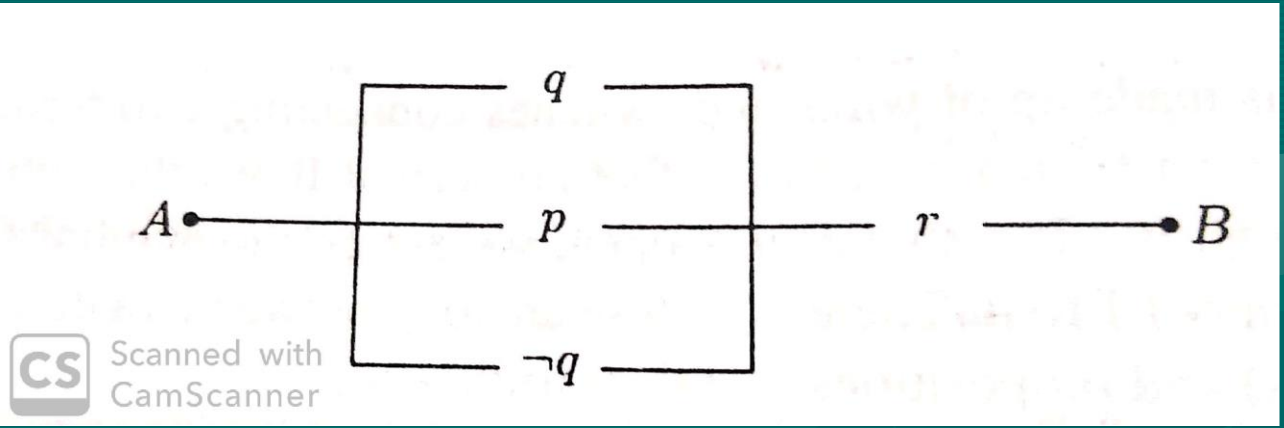
- This is a series network consisting of two switches p and q in which the current flows from the terminal T1 to T2 only if both the switches p and q are closed.
- Therefore this network ca n be represented by $p \wedge q$.

Switching Networks..

- Any switching network is a combination of any of these three types of networks.
- By applying the laws of logic on switching network, we can obtain a new network which are equivalent to the given network, may be with fewer number of switches

Examples:

Simplify the switching network shown below:



The above network can be represented by

$(q \vee p \vee \neg q) \wedge r$

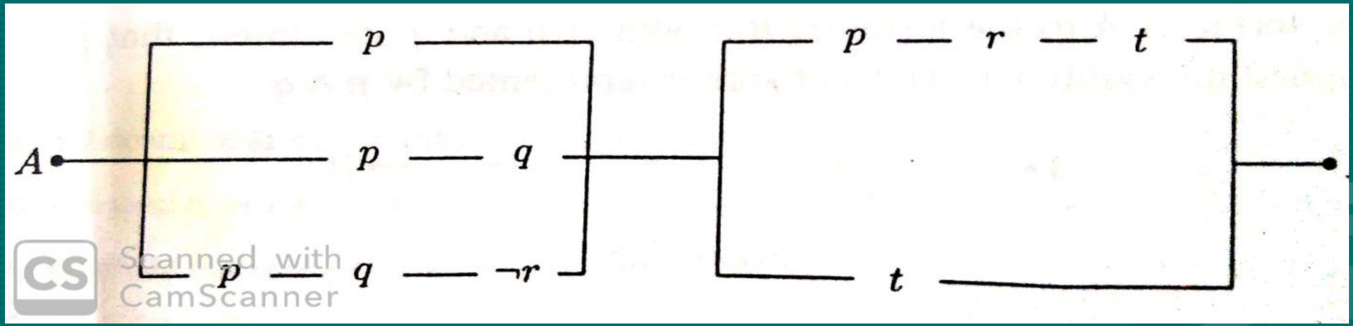
Solution

- $(q \vee p \vee \neg q) \wedge r$ commutative Law
- $(q \vee \neg q \vee p) \wedge r$ Inverse Law
- $T \wedge r$ Identity Law
- r
- This shows that the given network with 4 switches is equivalent to a network that contains only one switch.

T1 _______________ r _______________T2

Example 2…

- Simplify the network shown below



- [p v (p ∧ q) v (p ∧q ∧¬r)] ∧ [(p ∧ r ∧ t) v t]

Solution :

40 $[p \vee (p \wedge q) \vee (p \wedge q \wedge \neg r)] \wedge [(p \wedge r \wedge t) \vee t]$

where $x = p \vee (p \wedge q) \vee (p \wedge q \wedge \neg r)$ and $y = (p \wedge r \wedge t) \vee t$.

Consider the expression x.

$[p \vee (p \wedge q) \vee (p \wedge q \wedge \neg r)]$

$= p \vee (p \wedge q)$ Absorption Law.

$= p$ Absorption Law.

Consider y :-

$[(p \wedge r \wedge t) \vee t]$

$= t \vee (t \wedge (p \wedge r))$ – Using Commutative & associative law.

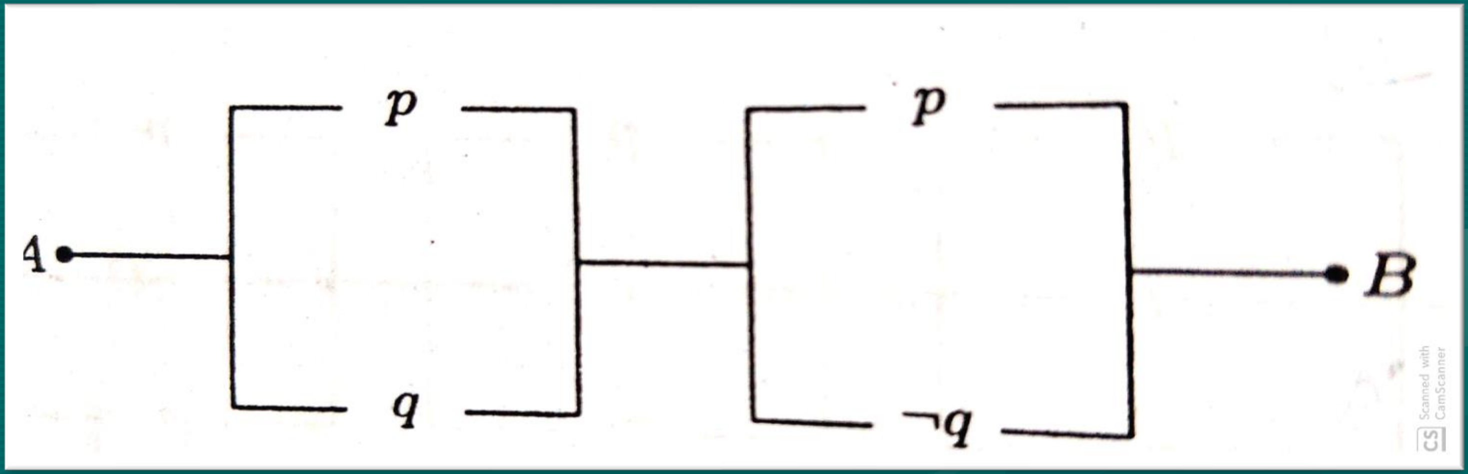
$= t$ – Absorption law.

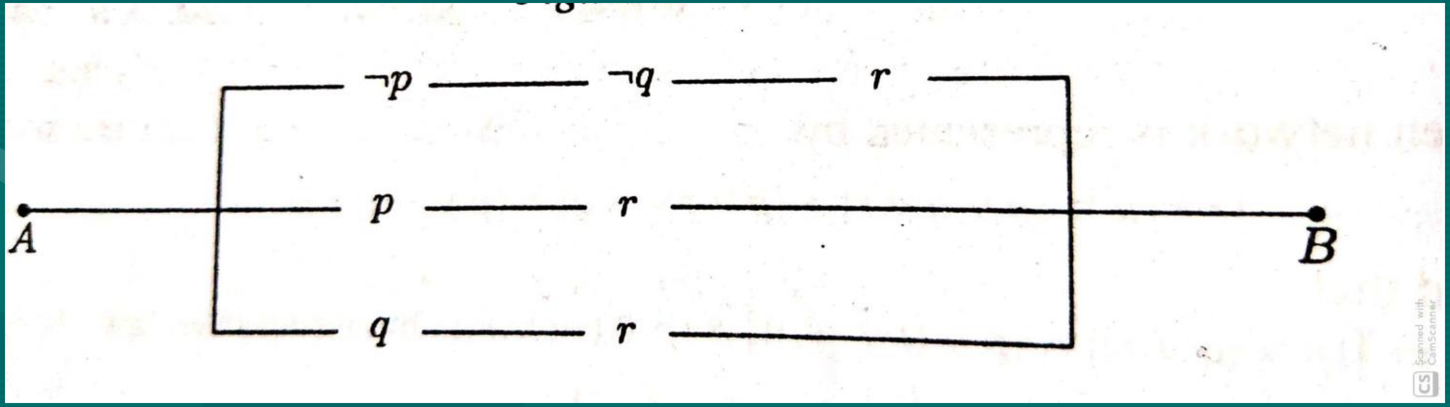
After simplification.

$= p \wedge t$

therefore the Number of switches has been reduced from 10 to 2 switches after simplification.

Example 3 & 4





Solution 3

Solution 3:

$(p \vee q) \wedge (p \vee \neg q)$

$= p \vee (q \wedge \neg q)$ – Distribution Law

$= p \vee F_0$ – Inverse law.

– Identity Law.

$= \underline{\underline{p}}$

Solution 4:

ution 4 :

$(\neg p \wedge \neg q \wedge r) \vee (p \wedge r) \vee (q \wedge r)$

$= (\neg p \wedge \neg q \wedge r) \vee (r \wedge (p \vee q))$ – distribution law.

$= r \wedge (\neg p \wedge \neg q) \vee (p \vee q)$ – distribution law

$= r \wedge \neg (p \vee q) \vee (p \vee q)$ – ~~negation law~~ Demorgan's law

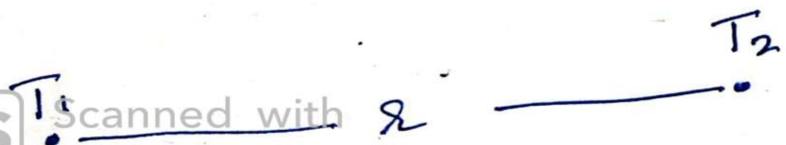
– Inverse law.

$= r \wedge T_0.$

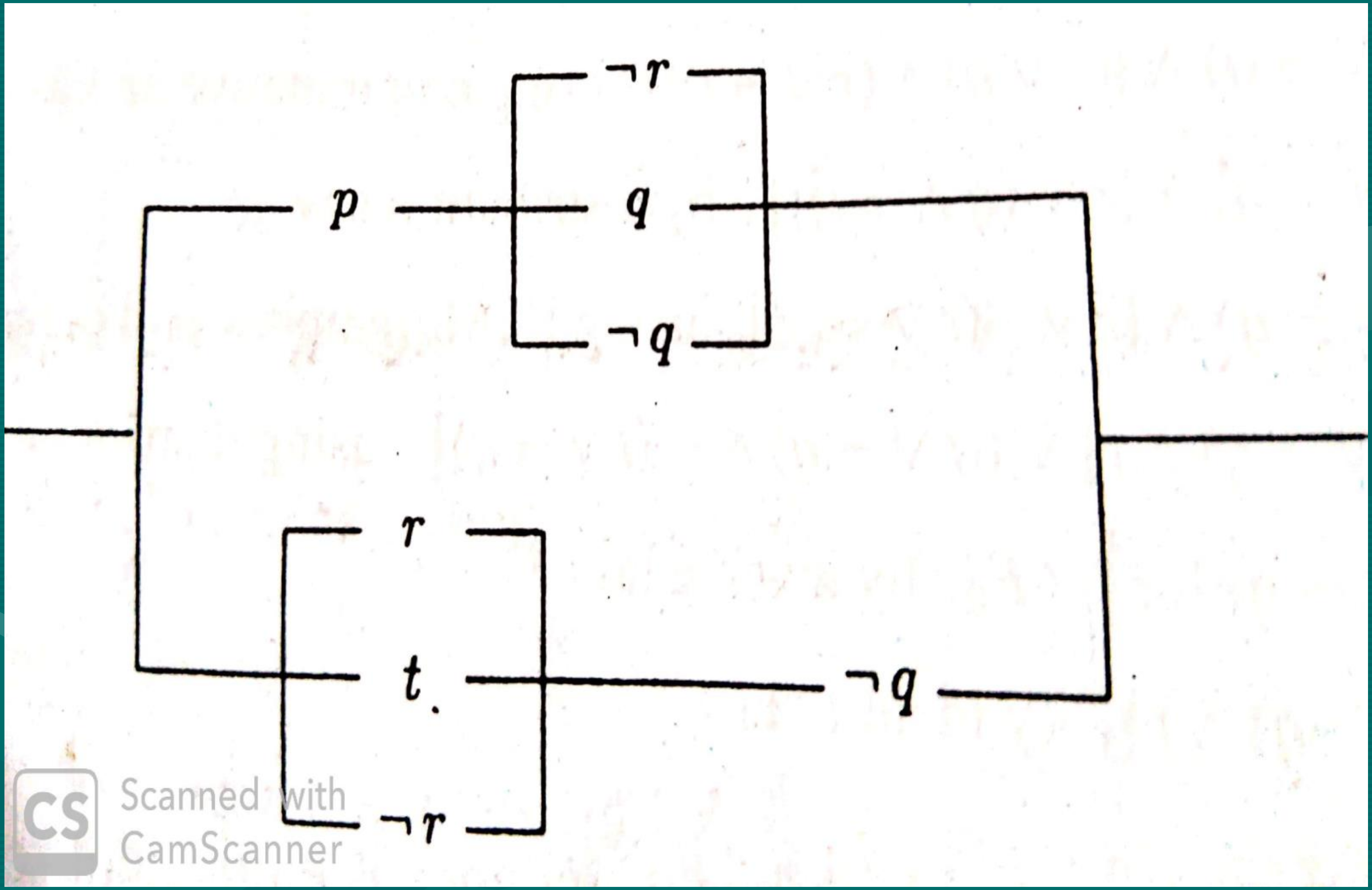
– Identity Law.

$= r$

No. of switches reduced to 1 from 7.



Try It Yourself



Duality

- Let s be a statement. If s contains no logical connectives other than ∧ and ∨, then the dual of s, denoted s^d, is the statement obtained from s by replacing each occurrence of ∨ by ∧ and ∧ by ∨ respectively, and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.)

- Eg. s : (p ∧ ¬q) ∨ (r ∧ T_0)
- s^d : (p ∨ ¬q) ∧ (r ∨ F_0)

Write down the duals for the following propositions :

1. $p \rightarrow q$

   $= [\neg p \vee q]^d$

   $= \neg p \wedge q$

2. $(p \rightarrow q) \rightarrow r$

   $= \neg(p \rightarrow q) \vee r$

   $= \neg(\neg p \vee q) \vee r$

   $= [p \wedge \neg q \vee r]^d$

   $= p \vee \neg q \wedge r$

3. $p \leftrightarrow q$?

Principle of Duality

Let s and t be the statements that contain no logical connectives other than ∨ and ∧ .

If $s \Leftrightarrow t$

then $s^d \Leftrightarrow t^d$

Example

Verify the principle of duality for the following logical equivalence.

38

$[\neg(p \wedge q) \rightarrow \neg p \vee (\neg p \vee q)] \Leftrightarrow (\neg p \vee q)$

Let $u = [\neg(p \wedge q) \Rightarrow \neg p \vee (\neg p \vee q)]$ – Implication law

$= [\neg(\neg(p \wedge q)) \vee \neg p \vee (\neg p \vee q)]$ – law of double negation

$= (p \wedge q) \vee \neg p \vee (\neg p \vee q)$ – Associative law.

$= (p \wedge q) \vee (\neg p \vee \neg p) \vee q$ – Idempotent law

$= (p \wedge q) \vee (\neg p \vee q)$ – distributive law.

$= [(p \wedge q) \vee \neg p] \vee ((p \wedge q) \vee q)$ – Absorption law

$= [(p \wedge q) \vee \neg p] \vee q$ – distributive law

$= [(p \vee \neg p) \wedge (q \vee \neg p)] \vee q$ ← Inverse law.

$= [T_0 \wedge (q \vee \neg p)] \vee q$ ← Identity law.

$= (q \vee \neg p) \vee q$ ← Associative law.

$= q \vee q \vee \neg p$ – Idempotent law.

$= q \vee \neg p$ – Commutative law

$= \neg p \vee q$

$u^d = [\neg p \vee q]^d = \neg p \wedge q.$

$v^d = [\neg p \vee q]^d = \neg p \wedge q.$

Prove that $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow p \wedge q$

Let $u = (\neg p \vee q) \wedge ((p \wedge p) \wedge q))$ — associative law

$= (\neg p \vee q) \wedge ((p \wedge p) \wedge q))$ — Idempotene law

$= (\neg p \vee q) \wedge (p \wedge q)$ — distributive law.

$= ((\neg p \vee q) \wedge p) \wedge ((\neg p \vee q) \wedge q)$ — Absorption law.

$= ((\neg p \vee q) \wedge p) \wedge q$ — distributive law

$= [(\neg p \wedge p) \vee (q \wedge p)] \wedge q$ — Inverse law

$= [F_0 \vee (q \wedge p)] \wedge q$ — Identity law

— associative law

$= (q \wedge p) \wedge q$ — Idempotene law

$= (q \wedge q \wedge p)$ — Commutative law

$= q \wedge p$

$u = p \wedge q$

$u^d = [p \wedge q]^d = p \vee q$ $\therefore u^d \Leftrightarrow v^d$

$v = [p \wedge q]^d$ $v^d = p \vee q$

Other Connectives : $\uparrow \downarrow$

For any two propositions p and q, the DeMorgan's law states that

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q \quad \text{and}$$

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

- The compound proposition $\neg(p \wedge q)$ is read as "Not of p and q" is also denoted by **$(p \uparrow q)$**
- The compound proposition $\neg(p \vee q)$ is read as " Not of p or q" and is also denoted by **$(p \downarrow q)$**

- The symbol ↑ is called the NAND connective and is the combination or *not* and *and* .
- The symbol ↓ is call the NOR connective and is the combination of *not* and *or*

p	q	¬p	¬q	¬pv¬q	¬(p ∧ q) ↑	
0	0	1	1	1	1	
0	1	1	0	1	1	
1	0	0	1	1	1	
1	1	0	0	0	0	

Thus, **(p ↑ q)** ⇔ ¬(p ∧ q) ⇔ ¬p ∨ ¬q and

(p ↓ q) ⇔ ¬(p ∨ q) ⇔ ¬p ∧ ¬q

Exercise problems:

For any proposition p and q, prove the following

1. $\neg(p \downarrow q) \Leftrightarrow \neg p \uparrow \neg q$

$$\neg(p \downarrow q) \Longleftrightarrow (\neg p \uparrow \neg q)$$

$$= \neg(\neg(p \vee q))$$

$$= \neg(\neg p \wedge \neg q)$$

$$= \neg p \uparrow \neg q$$

$$\therefore \neg(p \downarrow q) \Longleftrightarrow \neg p \uparrow \neg q$$

2. $\neg(p \uparrow q) \Leftrightarrow \neg p \downarrow \neg q$

$\neg(p \uparrow q) \Longleftrightarrow \neg p \downarrow \neg q$

$= \neg(\neg(p \wedge q))$ ← demorgan

$= \neg(\neg p \vee \neg q)$

$= \neg p \downarrow \neg q$

$\therefore \neg(p \uparrow q) \Longleftrightarrow \neg p \downarrow \neg q$

Are ↑ and ↓ Associative ?

Is

$p \uparrow (q \uparrow r) \Leftrightarrow (p \uparrow q) \uparrow r$?

No they are not .

No!!

34 $p \uparrow (q \uparrow r)$

$= p \uparrow (\neg(q \wedge r))$

$= \neg(p \wedge (\neg(q \wedge r)))$ - DM

$= \neg p \vee \neg(\neg(q \wedge r))$

$= \neg p \vee (q \wedge r)$

$(p \uparrow q) \uparrow r$

$= \neg(p \wedge q) \uparrow r$

$= \neg((\neg(p \wedge q)) \wedge r)$

$= (p \wedge q) \vee \neg r$

Since $p \uparrow (q \uparrow r) \not\Leftrightarrow (p \uparrow q) \uparrow r$.

hence $\uparrow$ is not associative.

Similarly with $\downarrow$ (NOR).

Exercise problems:

Express the following propositions in terms of only NAND and NOR connectives

1. $\neg p$
2. $p \wedge q$
3. $p \vee q$
4. $p \rightarrow q$

1. Prove that $(p \uparrow q) \Leftrightarrow (q \uparrow p)$, and $(p \downarrow q) \Leftrightarrow (q \downarrow p)$.

2. Prove that $[p \to (\neg p \to q)] \Leftrightarrow [p \uparrow (p \downarrow q)]$.

3. Prove the following:

(i) $(p \uparrow q) \Leftrightarrow (\neg p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$.

(ii) $(p \downarrow q) \Leftrightarrow (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$.

Converse, Inverse and Contrapositive: Logical Implication

Consider a conditional $p \rightarrow q$ then,

1. $q \rightarrow p$ is called the Converse of $p \rightarrow q$
2. $\neg p \rightarrow \neg q$ is called the Inverse of $p \rightarrow q$
3. $\neg q \rightarrow \neg p$ is called the Contrapositive of $p \rightarrow q$

State conditional , converse, inverse and contrapositive for the following primitive propositions

p: It rains

q: the game will be cancelled

1. $p \rightarrow q$: If it rains, then the game will be cancelled.
2. $q \rightarrow p$: If the game is cancelled, then it has rained.
3. $\neg p \rightarrow \neg q$: If it doesnot rain, then the game will not be cancelled.
4. $\neg q \rightarrow \neg p$: If the game is not cancelled, then it has not rained.

Truth Table :

p	q	¬p	¬q	p → q	q → p	¬p→¬q	¬q→¬p
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

From the above Truth Table we have the following important results:

1. The conditional and Contrapositive are logically equivalent for any proposition p and q

$$(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$$

2. The converse and Inverse of a conditional are logically equivalent for any proposition p and q.

$$(q \rightarrow p) \Leftrightarrow (\neg p \rightarrow \neg q)$$

And also $(p \rightarrow q) \not\Leftrightarrow (q \rightarrow p)$

$$(\neg p \rightarrow \neg q) \not\Leftrightarrow (\neg q \rightarrow \neg p)$$

Logical Implication :

Consider the implication p→q, for any two propositions p and q.

p : 6 is a multiple of 2

q : 3 is a prime number

Here p→q means " If 6 is a multiple of 2, then 3 is a prime number". Here since p is true and q is also true, p→q is also true, but this p→q does not make any sense because there no consistency in the statement p→q (though it is logical true)

- Hence we are interested in conditional $p \rightarrow q$ where p and q are related in some way, so that the truth value of q depends upon truth value of p or vice verse. Such conditional are called Hypothetical statements or Implicative statements.
- When a hypothetical statement $p \rightarrow q$ is such that q is true whenever p is true , then we say that p logically implies q and symbolically written as $p \Rightarrow q$.
- When a hypothetical statement $p \rightarrow q$ is such that q is is not necessarily true whenever p is true , then we say that p does not logically implies q and symbolically written as $p \not\Rightarrow q$.

Necessary and Sufficient Conditions :

Suppose that p → q , then in order that q may be true if is sufficient that p is true, also if p is true , then it necessary that q is true.

i.e For p → q   p is sufficient for q

q is necessary for p

Problems:

Using the truth table prove the following logical implication.

1. [p ∧ (p→q)] → q

p	q	p→q	p ∧ (p→q)
0	0	1	0
0	1	1	0
1	0	0	0
1	1	1	1

2. [(p→q) ∧ ¬q] → ¬q

p	q	p→q	¬q	p→q) ∧ ¬q
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	0	0

3. The following are 3 valid arguments. Establish the validity of each by the means of Truth Table. In each case, determine which rows of the table are crucial for accessing the validity of the argument and which rows can be ignored.

a. $[\, p \wedge (p \rightarrow q) \wedge r \,] \Rightarrow [\, (p \vee q) \rightarrow r \,]$

(a) (b)

p	q	r	p → q	(a)	p v q	(b)
0	0	0	1	0	0	1
0	0	1	1	0	0	1
0	1	0	1	0	1	0
0	1	1	1	0	1	1
1	0	0	0	0	1	0
1	0	1	0	0	1	1
1	1	0	1	0	1	0
1	1	1	1	1	1	1

b. $[[(p \wedge q) \rightarrow r] \wedge \neg q \wedge (p \rightarrow r)] \Rightarrow (\neg p \vee \neg q)$

c. $[[p \vee (q \vee r)] \wedge \neg q] \Rightarrow (p \vee r)$

Solution b : Rows 1,2 and 6

Solution c : Rows 2,5 and 6

Rules of Inferences

Establishing the validity of an argument by constructing the Truth Table is tedious method for the following reasons.

- Table size depends on the no. of primitives
- For establishing the validity we need only those rows whose truth value is true.

Solution : Truth Table method can be avoided for proving the validity of an argument by using the technique called Rules of Inference.

Rules of Inference are already valid arguments which can be used as templates for establishing the validity of other arguments

Rules of Inferences

Logical Implication: Rules of Inference

an argument: $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$

premises conclusion

is a valid argument

$\longleftrightarrow$ $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$ is a tautology

Rule 1 : Modus ponens / rule of detachment/method of affirming

In symbolic form of this rule is expressed by the logical implication.

$[p \wedge (p \rightarrow q)] \Rightarrow q$ [Symbolic form]

Can also be written is tabular form as below

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

This means that if p is true and $p \rightarrow q$ is true , then q must be true

Rule 1, can be verified using the truth table

p	q	p→q	p ∧ p→q	[p ∧ p→q] → q
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Since [p ∧ p→q] → q is a Tautology, therefore its proved

Example 1: Modus Ponens

Check the validity of the following argument.

p1 : *If Sachin hits a century, then he gets a free car.*

p2 : *Sachin hits a century.*

$\therefore$ *Sachin gets a free car*

Let

p: Sachin hits a century

q: Sachin gets a free car

Using this primitives we can write the arguments

$$p \rightarrow q$$

$$p$$

$\therefore q$ It's a valid argument ,as per Modus Ponens Rules

Example 2: Modus Ponens

P1 : If Sachin hits a century, then he gets a free care

P2 : Sachin does not get a free car

$\therefore$ *Sachin has not hit a century*

Let

p: Sachin hits a century

q: Sachin gets a free car

Using this primitives we can write the arguments

$$\begin{array}{r} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

It's a valid argument ,as per Modus Ponens

Rules

Example 3: Modus Ponens

P1 : If Sachin hits a century, then he gets a free care

P2 : Sachin does get a free car

$\therefore$ *Sachin has bit a century*

Let

p: Sachin hits a century

q: Sachin gets a free car

Using this primitives we can write the arguments

$$\begin{array}{r} p \rightarrow q \\ q \\ \hline \therefore p \end{array}$$

It's a Not a valid argument

$[(p \rightarrow q) \land q] \rightarrow p$ is not a tautology, hence not valid

Example 3: Modus Ponens

P1 : If Sachin hits a century, then he gets a free care

P2 : Sachin does get a free car

$\therefore$ *Sachin has hit a century*

Let

p: Sachin hits a century

q: Sachin gets a free car

Using this primitives we can write the arguments

$$p \rightarrow q$$

$$q$$

$\therefore p$ It's a Not a valid argument

$[(p \rightarrow q) \land q] \rightarrow p$ is not a tautology, hence not valid

Rule 2 : Modus Tollens / Method of denying

$$[(p \to q) \wedge \neg q] \to \neg p$$

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Example 1 : Modus Tollens

If I drive to work, then I will arrive tired.

I was not tired when arrived to work

∴ I didn't drive to work.

Let

p : I drive to work

q: I arrive tired

$$\begin{array}{rl} & p \rightarrow q \\ & \neg q \\ \hline \therefore & \neg p \end{array}$$

Is a Valid Argument

- Rule 3 : Rule of Syllogism

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$$

Example 1 : Rule of Syllogism

If I study, then I will not fail in the examination

If I don't fail in the examination, then my father gifts a two wheeler to me.

∴ *If I study, then my father gifts a two wheeler to me.*

Let

p : I study

q: I don't fail in the examination

r: my father gits a two wheeler to me.

$$p \rightarrow q$$
$$q \rightarrow r$$
$$\therefore p \rightarrow r$$

It's a valid argument.

Example 2: Syllogism

If Ravi goes out with friends, then he will not study.

If Ravi does not study, his father becomes angry

His father is not angry

∴ *Ravi has not gone out with friends.*

Let

p: Ravi goes out with friends

q: Ravi does not study

r: His father gets angry

$$p \rightarrow q$$
$$q \rightarrow r$$
$$\neg r$$

$$p \rightarrow r \quad \text{(syllogism)}$$
$$\neg r$$

$$\therefore \neg p \quad \text{(Modus Tollens)}$$

Problems 1:

P1 : if I study, I will not fail in the examination.

P2: If I don't watch tv in the evening, I will study.

P3: I failed in the examination.

C : I must have watched tv in the evening

Let the primitives be

p : I study

q: I failed in the examination

r: I watch tv in the evening

Then the given arguments can be read as

$p \rightarrow \neg q$

$\neg r \rightarrow p$

q

Therefore r

38 $[(p \rightarrow \neg q) \wedge (\neg r \rightarrow p) \wedge q] \rightarrow r$.

This argument can be proved as follows:-

i) $p \rightarrow \neg q \Longleftrightarrow \neg\neg q \rightarrow \neg p$ $= q \rightarrow \neg p$. ∵ Conditional & Contrapositive logically eq

Similarly

ii) $\neg r \rightarrow p \Longleftrightarrow \neg p \rightarrow \neg\neg r$ $= \neg p \rightarrow r$. — " —

Therefore the argument can be written as.

$q \rightarrow \neg p$
$\neg p \rightarrow r$ } Rule of syllogism.
q

$= q \rightarrow r$. (syllogism)
q

$\therefore r$. (Modus ponens)

∴ This is a valid argument.

Problem 2:

I will get grade A in this course or I will not graduate.

If I don't not graduate, then I will join army

I got grade A

Therefore I will not join the army

Let the primitives be

p: I get grade A in tis course

q: I graduated

r: I join the army

Solution

36 argument can be written as.

$$\begin{array}{l} p \lor \neg q \\ \neg q \rightarrow r \\ p \\ \hline \therefore \neg r \end{array}$$

The above premises can be written as.

$$\left.\begin{array}{l} \neg p \rightarrow \neg q \\ \neg q \rightarrow r. \end{array}\right\} \quad \because p \lor \neg q \Longleftrightarrow \neg p \rightarrow \neg q.$$

$$\begin{array}{l} p \\ \hline \therefore \neg r \end{array}$$

$$\begin{array}{l} \neg p \rightarrow r \\ p \\ \hline \therefore \neg r \end{array} \quad \text{(syllogism)}$$

$$\begin{array}{l} \neg r \rightarrow p \\ p \\ \hline \therefore \neg r \end{array} \quad \text{(contrapositive)}$$

Since $((\neg r \rightarrow p) \land p) \rightarrow \neg r$ is not tautology.

$\therefore$ It is not a valid argument.

Problem 3:

Consider the following arguments

1) *Rita is baking a cake*
2) *If Rita is baking a cake, then she is not practicing her flute*
3) *If Rite is not practicing her flute, then her father will not buy her a car*
4) *Therefore Rita's father will not buy her a car.*

Primitives are :-

P : Rita is baking cake.
q : Rita is practicing her flute.
r : Her father will buy her a car.

The arguments can be written as.

p
$p \to \neg q$
$\neg q \to \neg r$

$\therefore \neg r$

validity can be established as follows.

Steps		Reasons
1)	p.	premise.
2)	$p \to \neg q$	premise.
3)	$\neg q$	from step 1 & 2 & Rule Modus po
4)	$\neg q \to \neg r$.	premise.
5)	$\therefore \neg r$	from step 3 & 4 & Mo ponens.

OR.

Steps		Reasons.
1)	$p \to \neg q$	premise.
2)	$\neg q \to \neg r$	premise.
3)	$p \to \neg r$	from steps 1 & 2 & Sr
4)	p.	premise.
		from step 3 & 4 &

Problem 4:

Show that the following argument is valid (for primitive statements p, r, s, t and u) and the conclusion $\neg p$

$p \rightarrow r$, $r \rightarrow s$, $t \vee \neg s$, $\neg t \vee u$, $\neg u$

Solution :

34 Solution :-

Steps	Reasons
1) $p \to r$	premise
2) $r \to s$	premise.
3) $p \to s$	from 1 & 2 & Syllogism.
4) $t \vee \neg s$	premise
5) $\neg s \vee t$	Commutative law, on step 4
6) $s \to t$	implication law on step 5
7) $p \to t$	from step 3 & 6 & syllogism.
8) $\neg t \vee u$	premise.
9) $t \to u$.	Apply implication rule on step 8
10) $p \to u$	from step 7 & 9 & Syllogism,
11) $\neg u$	premises.
12) $\therefore \neg p$	from step 10 & 11 & Modus toll

$\therefore$ It's a valid argument.

Rule 4 : Rule of Conjunction

- This rule arises from the fact that if p and q are true statements then $p \wedge q$ is also a true statement.
- The statements p and q may occur in the development of an argument as a given premises or may be the result that are derived from the premises in the earlier arguments.
- Under these circumstances the two statements p and q can be combined into their conjunction $p \wedge q$, and this new statement can be used in later steps .
- Rule of Conjunction can be written as

$$\begin{array}{ll} & p \\ & q \\ \hline \text{Therefore} & p \wedge q \end{array}$$

Problem on application of Rule of Conjunction

Test the validity of the following argument.

$$p \wedge q$$
$$p \rightarrow (q \rightarrow r)$$
$$\overline{\qquad\qquad}$$
$$r$$

33

Solution:-

Steps	Reasons
1) $p \wedge q$	Premise
2) $p \rightarrow (q \rightarrow r)$	Premise.
3) p	from step 1 & Rule of Conjunction
4) $q \rightarrow r$	from step 2 & 3 & Rule of modus ponens.
5) q	from step 1 & Rule of conjunction.
6) $\therefore r$.	from step 4 & 5 & pon

Rule 5 : Rule of Disjunctive Syllogism

$$[(p \vee q) \wedge \neg p] \rightarrow q$$

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline q \end{array}$$

We know that $p \vee q = \neg p \rightarrow q$

$$\therefore \begin{array}{c} \neg p \rightarrow q \\ \neg p \\ \hline \therefore q \end{array}$$ (Modus ponens)

Rule 6 : Rule of Contradiction

Let 'p' denote an arbitrary statement, and F_0 a contradiction, then

$(\neg p \rightarrow F_0) \rightarrow p$ i.e $\dfrac{\neg p \rightarrow F_0}{p}$

Rule 7 : Rule of Conjuctive Simplification

$[(p \Lambda q)] \rightarrow p$ i.e $\dfrac{p \Lambda q}{p}$

Rule 8 : Rule of Disjunctive Amplification

$p \rightarrow (p \vee q)$ i.e $\dfrac{p}{p \vee q}$

Problems :

1. Demonstrate the validity of the following argument

$$\begin{array}{c} p \rightarrow r \\ \neg p \rightarrow q \\ q \rightarrow s \\ \hline \neg r \rightarrow s \end{array}$$

32 Problems 1: Solution

Steps	Reasons
1. $p \to r$	premise.
2. $\neg r \to \neg p$	from step 1 and contrapositive.
3. $\neg p \to q$.	premise.
4. $\neg r \to q$	from 2 & 3 & Rule of Syllogism.
5. $q \to s$	premise.
6. $\therefore \neg r \to s$	from 4 & 5 syllogism.

2. Establish the validity of the argument

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow (r \land s) \\ \neg r \lor (\neg t \lor u) \\ p \land t \\ \hline u \end{array}$$

Problem 2.

Steps	Reasons
1. $p \to q$	Premise.
2. $q \to (r \wedge s)$	premise.
3. $p \to (r \wedge s)$	Step 1 & 2 & Syllogism.
4. $p \wedge t$	premise.
5. p	from step 4 & Rule of conjunction simplification
6. $r \wedge s$	Step 3 & 5 & Modus ponens
7. r	Step 6 & Rule of conj simplification
8. $\neg r \vee (\neg t \vee u)$	Premise.
9. $(\neg r \vee \neg t) \vee u$	step 8 & associative law.
10. $\neg (r \wedge t) \vee u$	step 9 & demorgan's law.
11. $\neg r \vee u$.	Step 10 & law of Conjunctive simplic
12. $r \to u$.	step 11 law of Implication
13. $\therefore u$	Step 7 & 12 & law of Modus ponen

Show that the following argument is valid

If the band could not play rock music or the refreshments were not delivered on time, then the New year's party would have been cancelled and Alicia would have been angry.

If the party were cancelled , then refunds would be made

No refunds were made

Therefore the band could play rock music.

Steps	Reasons
1) $r \to t$	premise
2) $\neg t$	premise.
3) $\neg r$	From 1 & 2 & Modus Tollens
4) $\neg r \vee \neg s$	From 3 & Rule of Disjunctive Amplification
4) $\neg (r \wedge s)$	From 4 & DML $p \Rightarrow p \vee q$
5) $(\neg p \vee \neg q) \to (r \wedge s)$	Premises
6) $\neg(\neg p \vee \neg q)$	From 4 & 5 & Modus Tollens
7) $\neg\neg(p \wedge q)$	From 6 & De'morgan's law.
8) $(p \wedge q)$	From 7 & Law of Double negation
9) p	From 8 & law of Conjunctive Simplification.

∴ this argument is valid.

Quanitifier

- A sentence that involves a variable , such as x, cannot be called as a statement.
- These cannot be called as proposition unless the valve of x is specified.
- Eg., x+2 is an even integer.

  x divides 16 etc.,
- Sentences of this kinds are called as Open Statements.
- If the open statement contains a variable ‘x’ and when we restrict certain allowable choice for this ‘x’. These allowable choices are called the Universe or Universe of Discourse for open statement.

Quantifier…

- Open statement containing a variable 'x' are denoted by p(x),q(x) etc.,
- If U is the universe for the variable x in an open statement p(x) and if a ∈ U, then the proposition got by replacing x by a in p(x) is denoted by p(a).

Eg., p(5) : 5 + 2 is an even integer (false)

¬p(7): 7 + 2 is not a even integer (True)

- q(x,y) is used to represent an open statement that contain two variables.

q(x,y) : "The numbers y+2,x-y and x+2y are even integers"

q(4,2) : "The numbers 4, 2 and 8 are even integers" (True)

q(4,1): "The numbers 3, 3 and 6 are even integers" (False)

Quantifier…

- In the examples of p(x) and q(x,y) some substitutions results in true statements and others results in false statement.
- Therefore we can say

  For some x, p(x) is True

  For some x,y , q(x,y) is True

Similarly,

  For some x, ¬p(x) is False

  For some x,y , ¬q(x,y) is False

- The phrases "For some x" and "For some x,y" are said to quantify the open statements p(x) and q(x,y) respectively

Quantifier…

- There are two type of Quantifiers
  - ❖ Existential Quantifiers (∃)
  - ❖ Universal Quantifiers (∀)

Existential Quantifier (∃)

- The phrase "For some" are called existential quantifiers, which can also be expressed as "For at least one x" or "There exists an x such that " etc.,
- This quantifier is symbolically denoted as ∃x, hence the statement "for some x , p(x)" can be written as $\exists x\ p(x)$
- Similarly, $\exists x\ \exists y\ \ q(x, y)$ or $\exists x, y, q(x, y)$

Quantifier…

Universal Quantifiers (∀)

- Are denoted by $\forall x$ and is read as "For all x " or "For any x" or "For each x" or "For every x"

  Eg., p(x) : "The number x + 2 is even integer"

- The above open statement can be to quantified statement $\forall x, p(x)$ is a false statement".

  Eg., r(x) : "2x is an even integer"

- Then the quantified statement for the above open statement $\forall x, r(x)$ is a True statement.

- Note that the quantified statement $\exists x\ r(x)$ is also a True Statement, but $\forall x, \neg r(x)$ and $\exists x, \neg r(x)$ is False

Quantifier…

- The variable x in p(x) is called as Free variable (∀x)
- The variable x in r(x) is called as Bounded Variable (∃x)

Quantifier and Logical Connectives

- Compound open statements can be formed using logical connectives.

Eg., $\neg p(x)$, $p(x) \wedge q(x)$, $p(x) \rightarrow q(x)$

Example :

Suppose that the Universe consists of all integers, consider the following open statements.

$p(x) : x \leq 3$

$q(x) : x + 1$ is odd

$r(x) : x > 0$

Quantifier and Logical Connectives…

Write the Truth value for the following.

i) $p(2)$

ii) $\neg q(4)$

iii) $p(-1) \wedge q(1)$

iv) $\neg p(3) \vee r(0)$

v) $p(0) \rightarrow q(0)$

vi) $p(1) \leftrightarrow \neg q(2)$

vii) $p(4) \vee [\, q(1) \wedge r(2)]$

viii) $p(2) \wedge [q(0) \vee \neg r(2)]$

Solution :

i. True

ii. False

iii. False

iv. False

v. True

vi. False

vii. False

viii. True

Converse, Inverse and Contrapositive for open statements

- Definition : For the given open statements p(x) and q(x) defined for prescribed universe and the universally quantified statement $\forall x \; [\, p(x) \rightarrow q(x) \,]$, then

$\forall x \; [\, \neg q(x) \rightarrow \neg p(x) \,]$ is the Contrapositive

$\forall x \; [\, q(x) \rightarrow p(x) \,]$ is the Converse

$\forall x \; [\, \neg p(x) \rightarrow \neg q(x) \,]$ is the Inverse

Exercise Problems:

- Let p(x),q(x) and r(x) denote the following open statement

  $p(x) : x^2 - 7x + 10 = 0$

  $q(x) : x^2 - 2x - 3 = 0$

  $r(x) : x < 0$

Determine the Truth or Falsity of the following statements, when the universe contains only the integers 2 and 5

i) $\forall x \; [\, p(x) \rightarrow \neg r(x) \,]$

ii) $\forall x \; [\, q(x) \rightarrow r(x) \,]$

iii) $\forall x \; [\, p(x) \rightarrow r(x) \,]$

Solution:

$q(5)$.

Solution:-

30 i) $\forall x\,[p(x) \rightarrow \neg q(x)]$

Universe of discourse $\{2, 5\}$.

consider $x = 2$

$[p(2) \rightarrow \neg q(2)]$

$p(x): x^2 - 7x + 10 = 0.$

$p(2): 2^2 - 7(2) + 10 = 0$

$4 - 14 + 10 = 0.$

$0 = 0.$ (True)

$q(x): x < 0$

$q(2): 2 < 0.$ (False.

$\therefore p(2) \rightarrow \neg q(2).$

$T \rightarrow \neg F = T \rightarrow T =$ **True** consider $x = 5$.

$[P(5) \rightarrow \neg Q(5)]$

$P(x) : x^2 - 7x + 10 = 0$

$P(5) : 5^2 + 7(5) + 10 = 0$

$: 25 + 35 + 10 = 0$

$: 0 = 0$ (True)

$Q(x) : x < 0$.

$Q(5) : 5 < 0$ (False)

$\therefore P(5) \rightarrow \neg Q(5)$.

$= T \rightarrow \neg F = T \rightarrow T =$ True.

Therefore $\forall x [P(x) \rightarrow \neg Q(x)$ is TRU

Truth values of Quantified Statements

- The statement " $\forall x \in S, p(x)$" is True only when p(x) is true for each $x \in S$.
- The statement "$\exists x \in S$, p(x)" is False only when p(x) is false for every $x \in S$
- Similarly, for the proposition of the form " $\forall x \in S, p(x)$" to be false, its enough to exhibit one element 'a' of 'S' such that p(a) is false. This element 'a' is called as **Counter Example**
- For the proposition of the form "$\exists x \in S$, p(x)" to be true it enough to exhibit one element 'a' of 'S' such that p(a) is true

Rules of Inference for Open Statement:

- If $p(x)$ is an open statement for a given universe, and if $\forall x\ p(x)$ is true, then $p(a)$ is true for each 'a' in the Universe. This is called Rule of **Universal Specification.**

- If $p(x)$ is an open statement and it is true for any arbitrary 'x' chosen from a set of 'S' then the statement $\forall x \in S\ p(x)$ is true. This is known as the Rule of **Universal Generalization**

Logical Equivalence

Two quantified statements are said to be logically equivalent, whenever they have same truth value in all possible situations

i. $\forall x, [p(x) \wedge q(x)] \Leftrightarrow (\forall x, p(x)) \wedge (\forall x, q(x))$

ii. $\exists x, [p(x) \wedge q(x)] \Leftrightarrow (\exists x, p(x)) \wedge (\exists x, q(x))$

iii. $\exists x, [p(x) \rightarrow q(x)] \Leftrightarrow \exists x, [\neg p(x) \vee (\exists x, q(x)]$

Rules of Negation of Quantified Statements

To Construct the negation of a quantified statement, change the quantifier from Universal to Existential and Vice-versa and also replace the open statement by its negation.

$$\neg\{\forall x, p(x)\} \equiv \exists x, \{\neg p(x)\}$$

$$\neg\{\exists x, p(x)\} \equiv \forall x, \{\neg p(x)\}$$

Problems:

Negate and Simplify each of the following:

1. $\exists x, [p(x) \lor q(x)]$
2. $\forall x, [p(x) \land \neg q(x)]$
3. $\forall x, [p(x) \rightarrow q(x)$
4. $\exists x, [\{p(x) \lor q(x)\} \rightarrow r(x)]$

Solution:

1. $\forall x, [\neg p(x) \wedge \neg q(x)]$
2. $\exists x, [\neg p(x) \vee q(x)]$
3. $\exists x, [p(x) \wedge \neg q(x)]$
4. $\forall x, [\{p(x) \vee q(x)\} \wedge \neg r(x)]$

Problem :

- Consider the following program segment

  for n=1 to 10

  A[n] = n * n – n

Write the following statements in quantified form

a. Every entry in the array is non-negative
b. There exit two consecutive entries in A where the larger entry is twice the smaller.
c. The entries in the array are sorted in ascending order

Solution

a. Every entry in the array is non-negative

$\forall n(A[n] \geq 0)$

b. There exit two consecutive entries in A where the larger entry is twice the smaller.

$\exists n\ (A[n+1] = 2A[n])$

c. The entries in the array are sorted in ascending order

$\forall n(A[n] < A[n+1])$

Validity of a Open Statement :

The Rule of universal specification indicates that the truth of an open statement in one particular instances follows (as a special case) from a more general (for the entire universe) truth of that universally quantified open statement

Eg., For the universe of all the people, consider the open statements

m(x) : x is a mathematics professor

c(x) : x has studied Calculus

- Now consider the following argument

  All mathematics professor have studied calculus

  Leo is a mathematics professor

  therefore Leo has studied calculus

Now, if ‘a’ represents this particular person ‘Leo’ (in the universe) then we can rewrite the above argument in symbolic form as

$$\forall x\ [m(x) \rightarrow c(x)]$$

$$m(a)$$

$$\therefore\ c(a)$$

So, to establish the validity of the given argument, we proceed as follows.

Steps	Reasons
1) $\forall x\, [m(x) \rightarrow c(x)]$	Premise.
2) $m(l)$	Premise.
3) $m(l) \rightarrow c(l)$	Step 1 and the Rule of Universal Specification
4 $\therefore c(l)$	from step 2 & 3 and Rules of Modus ponens.

Note:- the statements in step 2&3 are not quantified statements, hence we can apply the rules of inference to these statements to deduce the conclusion.

Problems

- Consider the universe of all triangles in the plane in conjunction with the open statement
- p(t) : t has two sides of equal length

  q(t) : t is an isosceles triangle

  r(t) : t has two angles of equal measures

The arguments are

In triangle xyz there is no pair of angles of equal measure

If a triangle has 2 sides of equal length, then it is isosceles

If a triangle is isosceles, then it has two angles of equal measure.

Therefore triangle xyz has no two sides of equal length.

- In symbolic form

$$\begin{array}{l} \neg r(c) \\ \forall t\,[\,p(t) \rightarrow q(t)] \\ \forall t[q(t) \rightarrow r(t)] \\ \hline \therefore \quad \neg p(c) \end{array}$$

Steps	Reasons
1) $\forall t\,[p(t) \rightarrow q(t)]$	Premise
2) $p(c) \rightarrow q(c)$	Step (1) and the Rule of Universal Specification
3) $\forall t\,[q(t) \rightarrow r(t)]$	Premise
4) $q(c) \rightarrow r(c)$	Step (3) and the Rule of Universal Specification
5) $p(c) \rightarrow r(c)$	Steps (2) and (4) and the Law of the Syllogism
6) $\neg r(c)$	Premise
7) $\therefore \neg p(c)$	Steps (5) and (6) and Modus Tollens

Therefore it's a Valid Argument

2. Check the validity of the following argument

Consider the universe to be the set of all quadrilateral and the open statements are

p(x) : x is square

q(x) : x has four sides

Arguments are

All square have four sides

The quadrilateral ABCD has four sides

Therefore ABCD is a square

In symbol forms

$$\forall x\,[p(x) \rightarrow q(x)]$$
$$q(a).$$
$$\therefore p(a)$$

Steps	Reasons
1. $\forall x[p(x) \rightarrow q(x)]$	premise.
2. $p(a) \rightarrow q(a)$	Step 1 & Rule of US
3. $q(a)$	premise.
4. $\therefore p(a)$	

$\therefore$ the above argument is not vali

Find whether the following argument is valid or not.(consider the all the engineering students as universe)

No engineering student of 1st or 2nd semester studies logic design

Anil is an engineering student who studies logic design

Therefore Anil is not in Second Semester.

- Open statements are
- p(x) : x is in first semester

  q(x) : x is in second semester

  r(x) : x studies logic design

Arguments in symbolic form:

$$\forall x, \left[\left(p(x) \vee q(x)\right) \rightarrow \neg r(x)\right]$$

$$\frac{r(a)}{\therefore \quad \neg q(a)}$$

Solution:

Steps	Reasons
1. $\forall x[(p(x) \vee q(x)) \rightarrow \neg r(x)]$	- premise
2. $[(p(a) \vee q(a)) \rightarrow \neg r(a)]$	- step 1 & law of US
3. $r(a) \rightarrow \neg(p(a) \vee q(a))$	- step 2 & contrapositive.
4. $r(a)$	premise.
5. $\neg(p(a) \vee q(a))$	- step 3 & 4 & ponens.
6. $\neg p(a) \wedge \neg q(a)$.	- step 5 & Demorgan law.
7. $\therefore \neg q(a)$	- step 6 & Conjunctive. simplification.

$\therefore$ The argument is valid

29

The Rule of Universal Generalization

- The rule states that if an open statement p(x) is proved to be true when x is replaced by any arbitrarily chosen element ‘ c’ from the universe, then the universally quantified statement $\forall x\, p(x)$ is also true

Consider the examples,

Let p(x),q(x) and r(x) be open statement, show that the following argument is valid

$$\forall x\,[p(x) \rightarrow q(x)]$$
$$\forall x\,[q(x) \rightarrow r(x)]$$
$$\overline{\therefore \forall x\,[p(x) \rightarrow r(x)]}$$

Solution:

Steps	Reasons
1. $\forall x[p(x) \rightarrow q(x)]$	Premise.
2. $p(c) \rightarrow q(c)$	Step 1 and Rule of Universa Specification
3. $\forall x[q(x) \rightarrow r(x)]$	Premise.
4. $q(c) \rightarrow r(c)$	Step 3 and Rule of Universal specificatio
5 $p(c) \rightarrow r(c)$	from Step 2 & 4 & Rules of Syllogism.
6. $\therefore \forall x[p(x) \rightarrow r(x)]$	from step 5 & Rules of Universal Generalization.

$\therefore$ valid argument

1.	$P \vee \sim P$	Law of the excluded middle
2.	$\sim (P \wedge \sim P)$	Contradiction
3.	$[(P \to Q) \wedge P] \to Q$	Modus ponens
4.	$[(P \to Q) \wedge \sim Q] \to \sim P$	Modus tollens
5.	$\sim\sim P \leftrightarrow P$	Double negation
6.	$[(P \to Q) \wedge (Q \to R)] \to (P \to R)$	Law of the syllogism
7.	$(P \wedge Q) \to P$ $(P \wedge Q) \to Q$	Decomposing a conjunction Decomposing a conjunction
8.	$P \to (P \vee Q)$ $Q \to (P \vee Q)$	Constructing a disjunction Constructing a disjunction
9.	$(P \leftrightarrow Q) \leftrightarrow [(P \to Q) \wedge (Q \to P)]$	Definition of the biconditional
10.	$(P \wedge Q) \leftrightarrow (Q \wedge P)$	Commutative law for $\wedge$
11.	$(P \vee Q) \leftrightarrow (Q \vee P)$	Commutative law for $\vee$
12.	$[(P \wedge Q) \wedge R] \leftrightarrow [P \wedge (Q \wedge R)]$	Associative law for $\wedge$
13.	$[(P \vee Q) \vee R] \leftrightarrow [P \vee (Q \vee R)]$	Associative law for $\vee$
14.	$\sim (P \vee Q) \leftrightarrow (\sim P \wedge \sim Q)$	DeMorgan's law
15.	$\sim (P \wedge Q) \leftrightarrow (\sim P \vee \sim Q)$	DeMorgan's law
16.	$[P \wedge (Q \vee R)] \leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$	Distributivity
17.	$[P \vee (Q \wedge R)] \leftrightarrow [(P \vee Q) \wedge (P \vee R)]$	Distributivity
18.	$(P \to Q) \leftrightarrow (\sim Q \to \sim P)$	Contrapositive
19.	$(P \to Q) \leftrightarrow (\sim P \vee Q)$	Conditional disjunction
20.	$[(P \vee Q) \wedge \sim P] \to Q$	Disjunctive syllogism
21.	$(P \vee P) \leftrightarrow P$	Simplification
22.	$(P \wedge P) \leftrightarrow P$	Simplification