Coding Theory

Introduction:

- In Digital communication system, information is transmitted in the form of Strings of 0’s and1’s
- While transmitted the information certain problems may raise due to noise in the transmission channel.
- Therefore, when certain signal is transmitted, a different signal may be received causing the receiver to make wrong decision.
- In this chapter we study about the techniques that help to detect and perhaps even to correct the transmission Errors.

Basics

- Consider a set $Z_2 = \{0,1\}$,
- then $Z_2^n = Z_2 \times Z_2 \times Z_2 \ldots$(n factors)
- $=\{(a1,a2,a3\ldots\ldots.an\}|\ a1,a2,\ldots\ldots an \in Z_2\}$
- Thus, every element of Z_2^n is a tuple (a1,a2,….an) in which every a_i belong to Z_2 so that it is a 0 or 1
- Eg., $Z_4^3 = (1,2,3)$ // elements in this group is with 4 and has 3 elements(3-tuple)
- $Z_4^5 = (2,3,3,1,1)$ // elements in this group is with 4 and has 5 elements (5-tuple)
- Similarly, the word/string 00101 is a 5-tuple $(0,0,1,0,1) \in Z_2^5$

Transmission of a word

- When a word c = c1c2c3……cn ∈ Z_2^n is transmitted from a point A through a transmission channel T.
- In ideal situation (if no noise) this word would be received at a point B without any changes.
- But in actual practice, transmission channels suffer disturbances called noise that may cause a ‘0’ to be transmitted a ‘1’ or vice versa.
- Hence a word ‘c’ transmitted from A is received as a different word ‘r’ ∈ Z_2^n at B

Transmission of word…

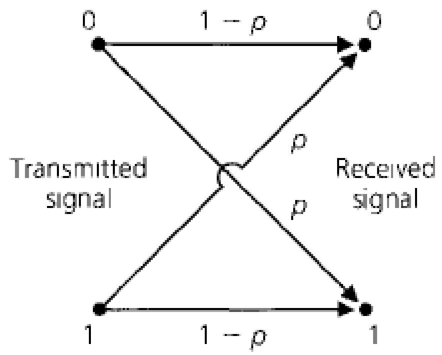
- The word ‘r’ will be of the form r = r1r2r3….rn where each r_i is a 0 or 1 but $r_j \neq c_j$ for some j, $1 \leq j \leq n$
- r can be written as r = c + e where c is a word in Z_2^n and e is error pattern

Symmetric Channel

- Suppose ‘P1’ is the probability(likelihood) that the transmitted signal 0 is received as the signal 1 and ‘P2’ is the probability that transmitted signal 1 is received as 0.
- The transmission channel is said to be symmetric if P1=P2.
- Thus, in Symmetric Channel the probability that a transmitted signal 1 is received as the signal 0 is equal to the probability that a transmitted signal 0 is received as 1.

Binary Symmetric Channel

- If p is the probability of incorrect transmission that a signal (0 or 1) is received correctly is 1-p



The Binary Symmetric Channel

- As we are transmitting only 0 or 1 (digital transmission) the symmetric channel is referred to as Binary Symmetric Channel

- Suppose ‘r’ differs from ‘c’ in exactly one place (say j^{th} place), then, $r1= c1, r2= c2, r3=c3, r_{j-1} = c_{j-1}$, $r_j \neq c_j$, $r_{j+1} = c_{j+1}$ ……$r_n = c_n$

  Since the probability that $r_i = c_i$ is (1-p) for each i, the probability

  that $r_1 = c_1$, $r_2=c_2$ …… $r_{j-1} = c_{j-1}$ is (By the product rule)

- $(1-p)(1-p)(1-p)(1-p) = (1 - p)^{j-1}$ // (j-1) factors
- Similarly the probability that , $r_{j+1} = c_{j+1}$ ……$r_n = c_n$

  $(1-p)(1-p)(1-p)(1-p) = (1 - p)^{n-j}$ // (n-j) factors
- Probability that $r_j \neq c_j$ is p

Continued….

- Therefore, the probability that

  $r1= c1,\ r2= c2, r3=c3,\ r_{j-1} = c_{j-1},\ r_j \neq c_j,\ r_{j+1} = c_{j+1} \ldots\ldots r_n = c_n$ is

  $= (1 - p)^{j-1} . p . (1 - p)^{n-j}$

  $= p. (1-p)^{n-1}$

This is probability that ‘r’ differs from ‘c’ in exactly one place, Similarly the probability that ‘r’ differs from ‘c’ in ‘k’ places is

$= p^k . (1-p)^{n-k}$

Problems:

1. The word c = 110101 is transmitted through a binary symmetric channel T. if the error pattern e=101010, find the word r=T(c) received.

Solution : c = 110101 $\in Z_2^6$

e = 101010 $\in Z_2^6$

r = c + e

= (1,1,0,1,0,1) + (1,0,1,0,1,0)

= (1+1, 1+0, 0+1, 1+0, 0+1,1+0)

r = 0 1 1 1 1 1 // (1+1 = 2-2) in Z_2

Problems..

2. The word c = 1010110 is transmitted through a binary symmetric channel. If e=0101101 is the error pattern, find the word r received, if p = 0.05 is the probability that a signal is incorrectly received, find the probability with which r is received.

Solution : Here the transmitted word c = 1010110 $\in Z_2^7$ and error pattern

e = 0101101 $\in Z_2^7$ and we know that r = c + e , where + is the addition

in Z_2^7

r = c+e =(1,0,1,0,1,1,0) +(0,1,0,1,1,0,1)

= (1+0 , 0+1, 1+0, 0+1, 1+1, 1+0, 0+1)

r = 1 1 1 1 0 1 1

Problems….

c = 1 0 1 0 1 1 0

r = 1 1 1 1 0 1 1

Comparing c and r, r differs from c in 4 places,

Therefore probability with which r is received is $= p^4 (1-p)^{7-4}$

$= (0.05)^4 (1-0.05)^3$

$= 0.000053$

Problems…

3. The word c=1010011 sent through a binary symmetric channel is received as the word r=T(c) = 1100110. Find the error pattern.

Solution : c=1010011 , r= 1100110

$r = c + e$

$1=1+e1$, Therefore $e1=0$

$1=0+e2$, $e2=1$

$0=1+e3$, $e3=1$

$0=0+e4$, $e4=0$

$1=0+e5$, $e5=1$

$1=1+e6$, $e6=0$

$0=1+e7$, $e7=1$ Therefore error pattern e = 0110101

Problems..

4. A word c transmitted through a binary symmetric channel is received as r=0000111. if e=0101111 is the error pattern , determine c.

Solution: $r = c + e$

$0=c1+0, \quad c1=0$

$0=c2+1, \quad c2=1$

$0=c3+0, \quad c3=0$

$0=c4+1, \quad c4=1$

$1=c5+1, \quad c5=0$

$1=c6+1, \quad c6=0$

$1=c7+1, \quad c7=0$ Therefore c=0101000

Problems..

5. The word c=1010110 is sent through a binary symmetric channel. If p=0.02 is the probability of incorrect receipt of a signal. Find the probability that c is received as r=1011111, determine error pattern.

6. A binary symmetric channel has probability p=0.05 of incorrect transmission . If the word c = 011011101 is transmitted, what is the probability that.

i) Single error occurs

ii) Double error occurs

iii) Triple error occurs

Solution:

The probability that exactly ‘k’ errors are made in the transmission is

$$nC_k\, p^k\, (1-p)^{n-k}$$